Backpropagation

Instructor: Mark Kramer

Part 1
Backpropagation

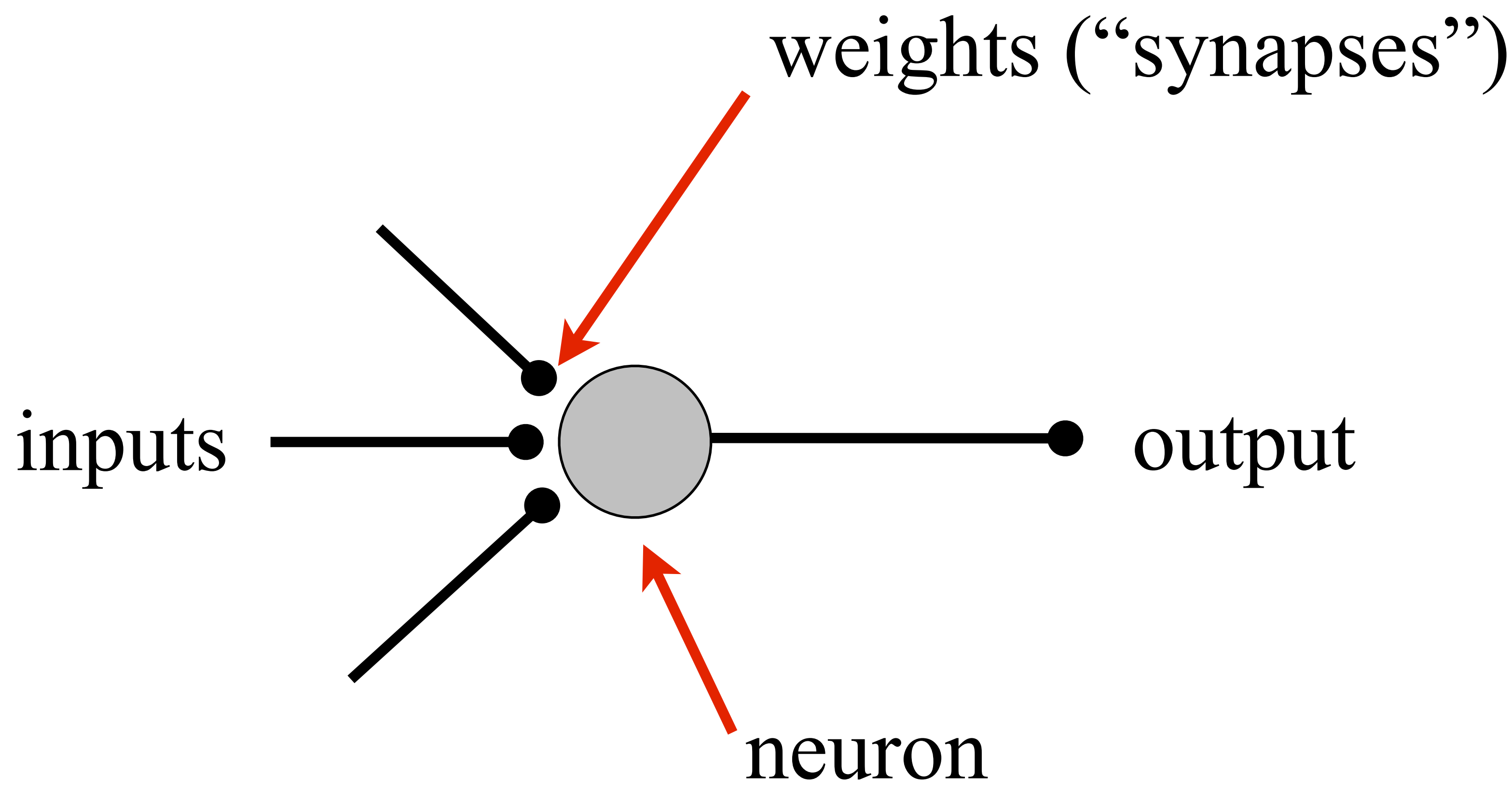
Today

We'll study learning in a "simple" neural network:

– Backpropagation

Remember, the Perceptron

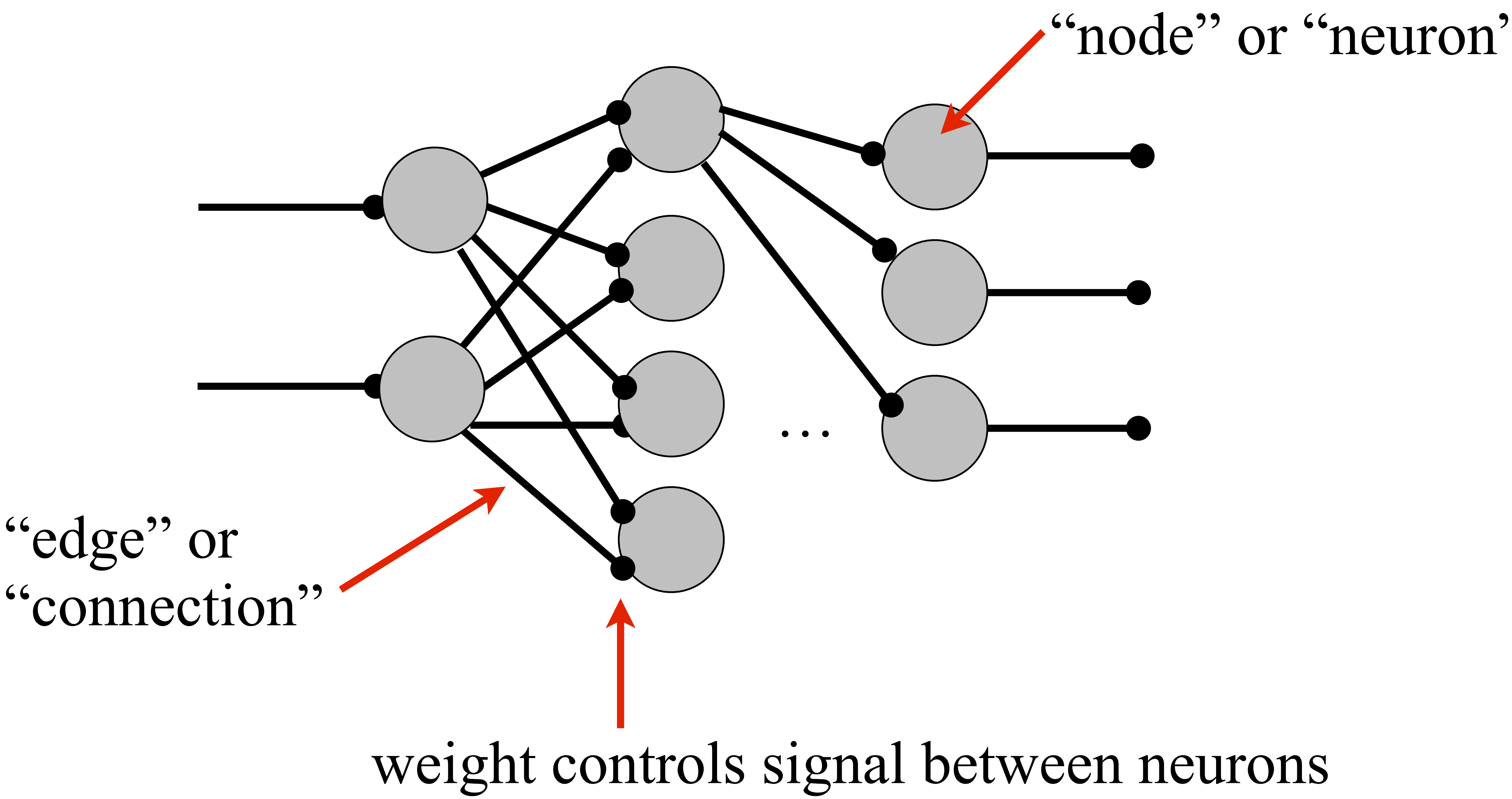
Cartoon & Cast of Characters



Note: there's only one neuron.

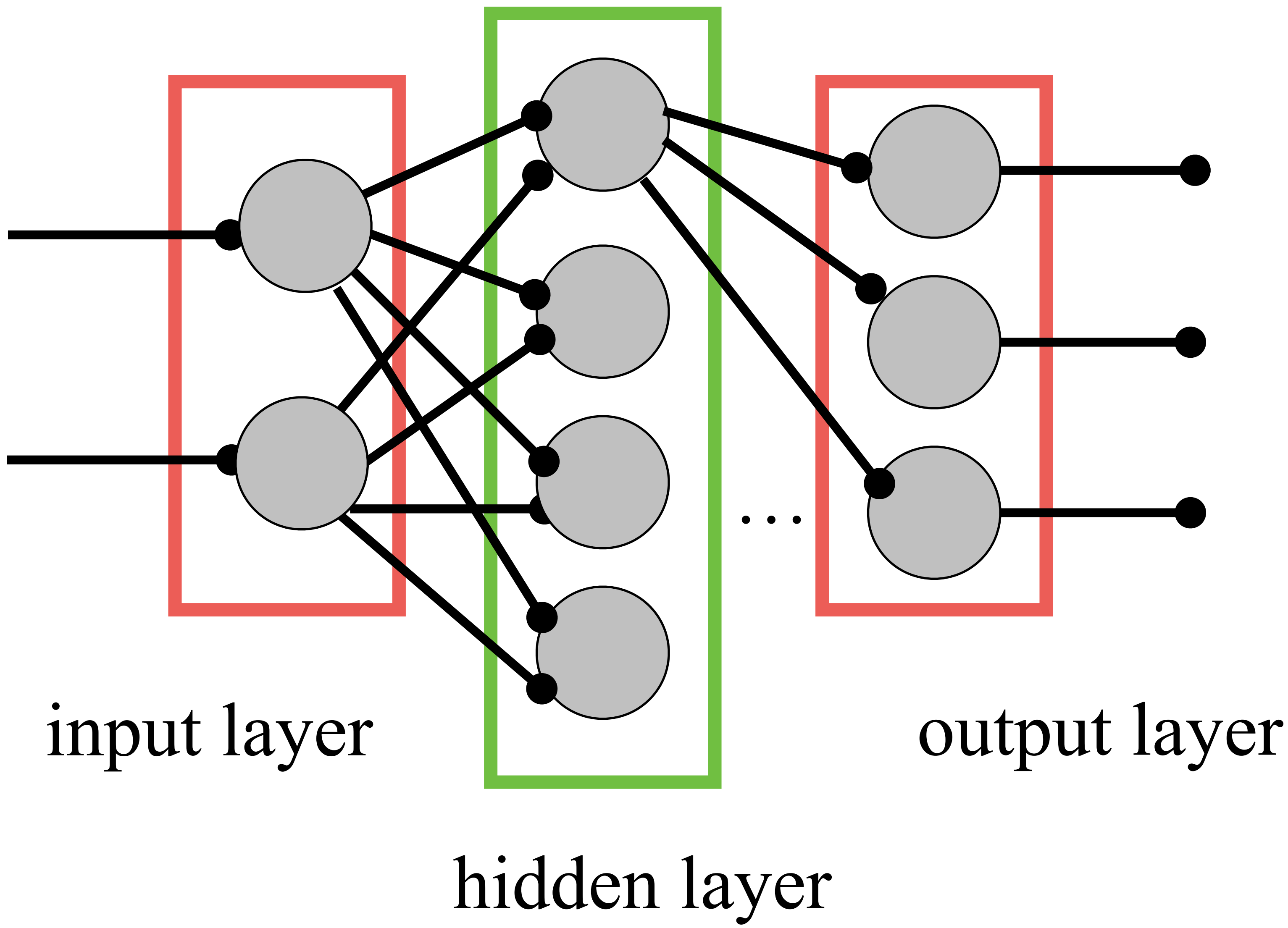
Today, a neural network

Today, **neural networks**: collections of abstracted neurons connected to each other through weighted connections (simplified “synapses”).



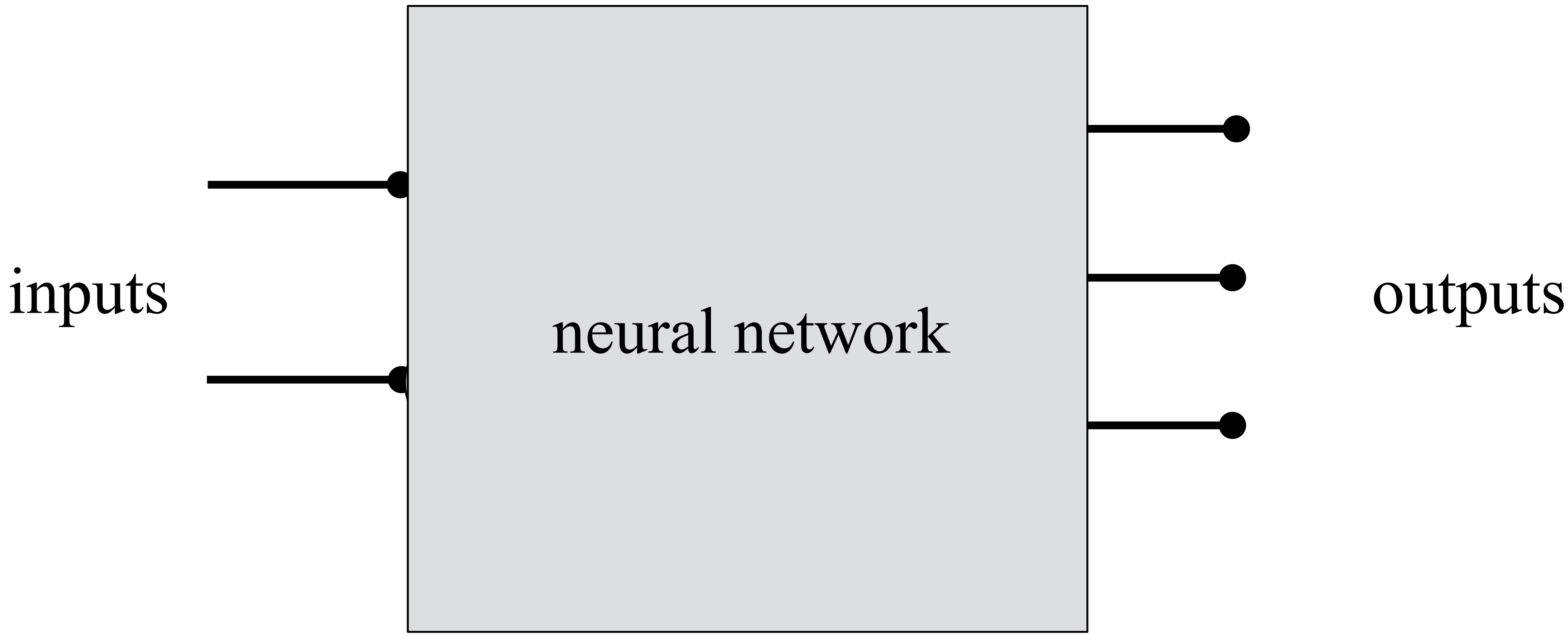
Today, a neural network

Today, **neural networks**: collections of abstracted neurons connected to each other through weighted connections (simplified "synapses").



Information processed through the network

Abstractly:



Neural networks can exhibit rich behavior.

Example: playground.tensorflow.org

Neural networks can learn

Neural networks are:

- **adaptive**
  - internal structure changes based on information flowing through the network.
- To do so, **adjust weights**.
- Idea:
  - When network outputs are “good”, preserve the weights.
  - When network output are “bad”, changes the weights.
    - When the network makes errors, adapt.

We trained a perceptron …

Now, we’ll train a neural network to do what we want …

Neural networks can learn

Some terminology:

- "*training a neural network*"
  calibrate weights to get output we want.

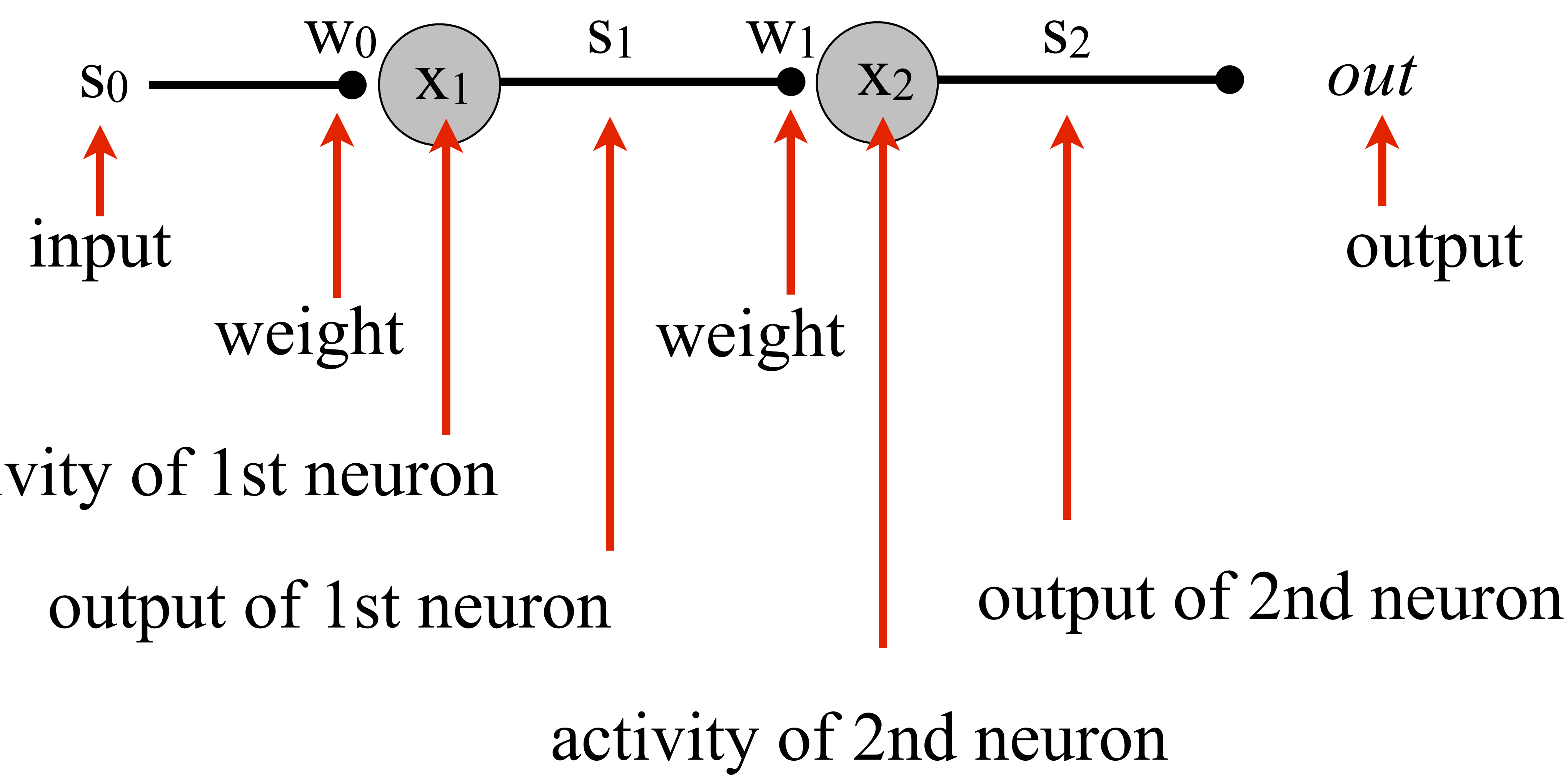
- **Forward propagation**
  For a set of weights & input, calculate output.

- **Backpropagation**
  Determine error in output, and adjust weights to decrease error.

Let's train a "simple" neural network to do something …

A “simple” neural network

Start with perceptron … add a node … and label everything.



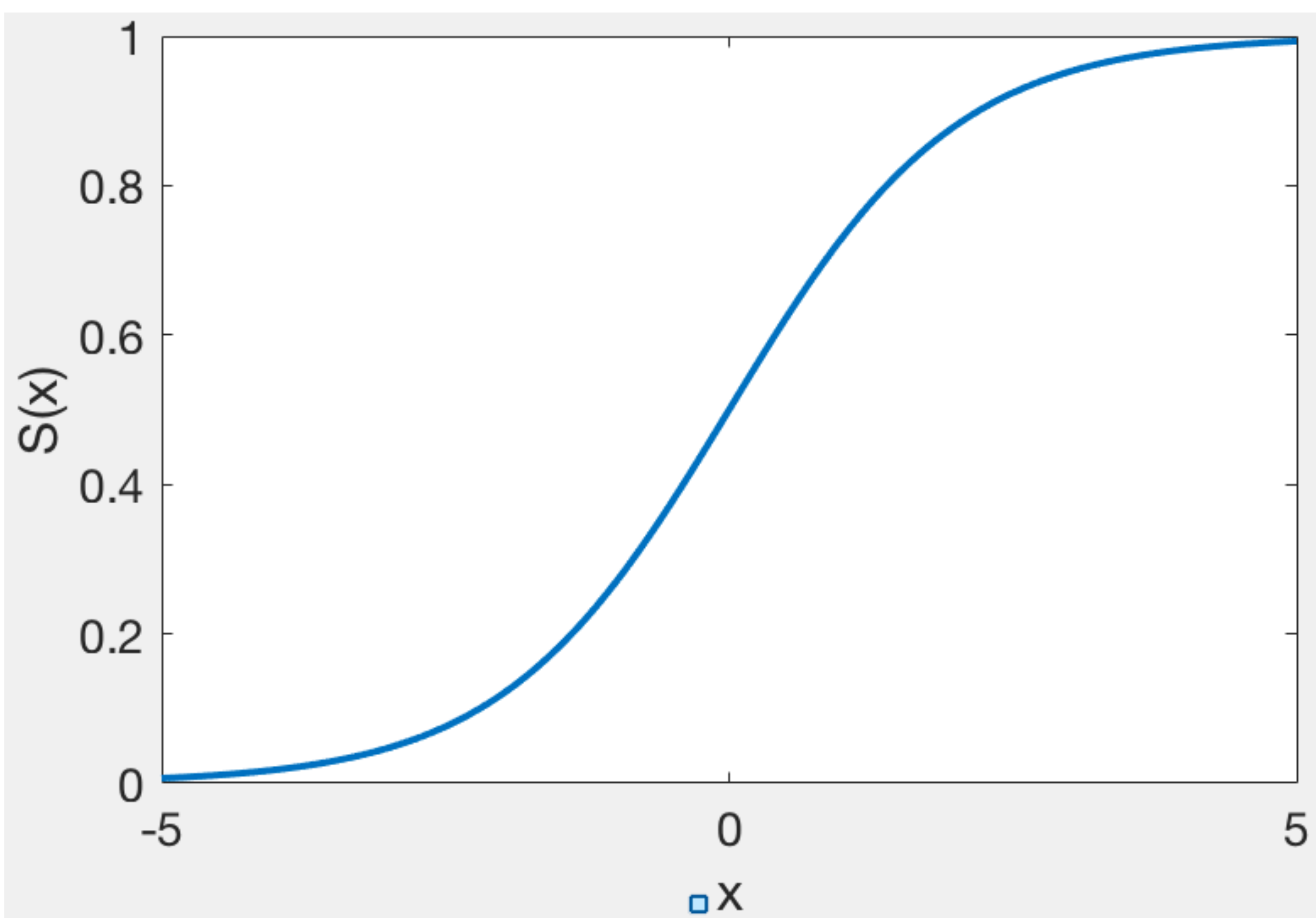
Activation function

Remember, the activation function:

x (activity) —activation function→ output

Here we'll use a **sigmoid** activation function:

$$S(x) = 1 / (1 + e^{-x})$$



NOTE: It's like a "smoothed" binary threshold.

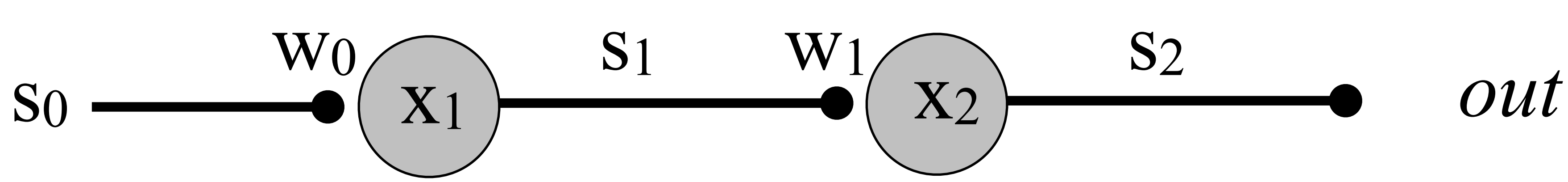
The challenge

We're given a list of inputs (s_0) and outputs (*out*):

s_0 *out*

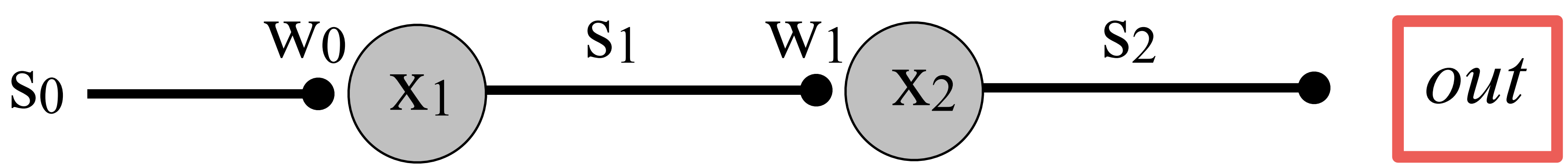
```
[[ 0.32625098  0.71507894]
 [-0.65595776  1.0065779 ]
 [-0.9664146   1.09676408]
 ...
 [ 1.53148642  0.48558244]
 [-1.07312405  1.12584863]
 [ 1.32680081  0.51212226]]
```

Q: What weights (w_0 w_1) produced these data?



A "simple" neural network

We want our network to **learn** …



so that when then input s_0=2, the input *out*=0.7

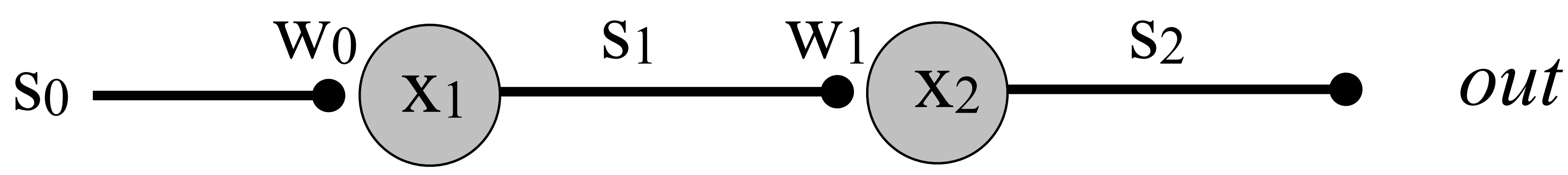
Q: How do we do it?

A: We need to choose the right weights: w_0 w_1

So, how do we find the right weights?

What are the right model weights?

Let's guess:



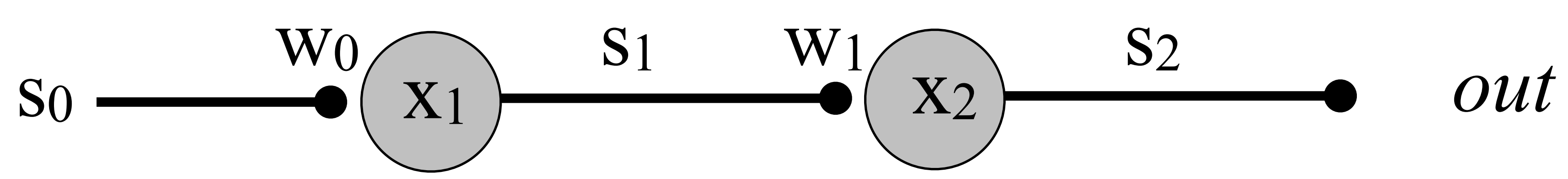
$w_0 = 2$ $\qquad$ $w_1 = 1$

Q: How did I choose these?

Q: Do they work?

A: Let's check … **forward propagation**

Forward propagation



$s_0 = 2$ $\quad w_0 = 2$ $\quad w_1 = 1$

target:
out = 0.7

Let's do it.

$$x_1 = w_0 s_0 = 2 * 2 = 4$$

$$s_1 = S(x_1) = S(4) = 0.982$$

$$x_2 = w_1 s_1 = 1 * 0.982 = 0.982$$

$$s_2 = S(x_2) = S(0.982) = 0.7275$$

$$out = s_2 = 0.7275$$

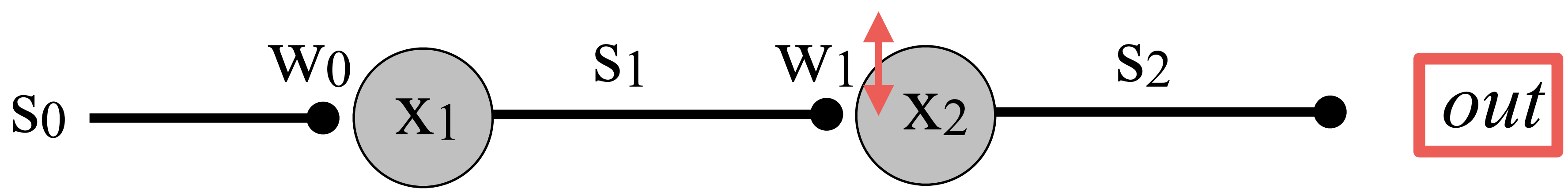
Match?

NO

How does a change in weight w_1 impact output?

Q: So now what?

<u>(intermediate) Goal</u>: get output (*out*) closer to target (0.7)



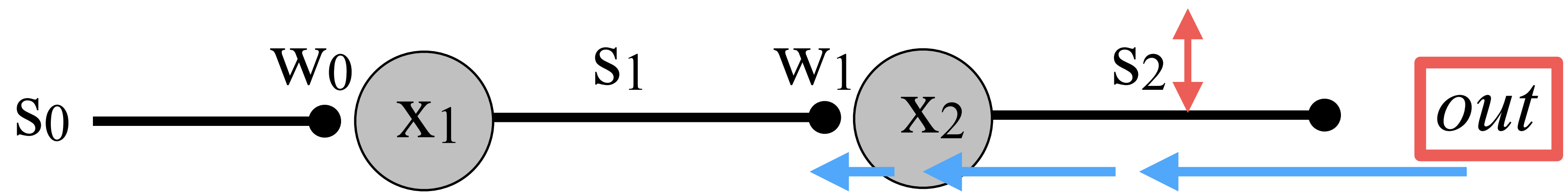
Q: How does a change in weight w_1 impact the output?

<u>Idea</u>: wiggle w_1 how does *out* change?

$$\frac{d\ out}{d\ w_1} = \text{Hmm} \ldots$$

out does <u>not</u> depend directly on w_1

How does a change in weight w_1 impact output?



out does depend on s_2 and s_2 depend on x_2 and x_2 depend on w_1

Mathematically … the **chain rule**

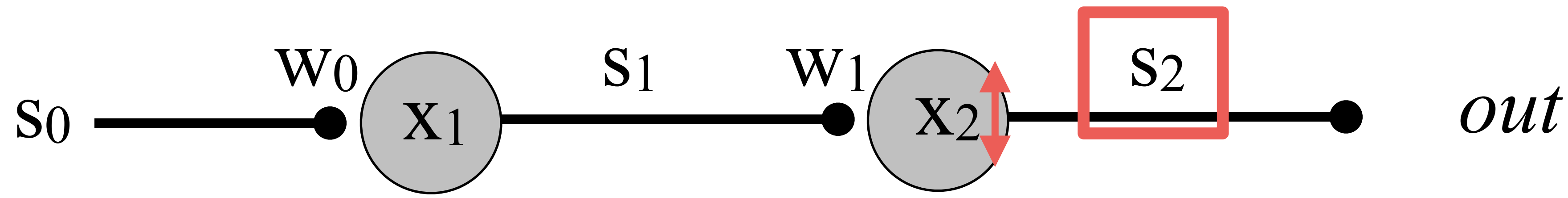
$$\frac{d\,out}{d\,w_1} = \boxed{\frac{d\,out}{d\,s_2}} \frac{d\,s_2}{d\,x_2} \frac{d\,x_2}{d\,w_1}$$

wiggle s_2 and *out* changes

Remember: $out = s_2$

$$\frac{d\,out}{d\,s_2} = \frac{d\,(\,s_2\,)}{d\,s_2} \boxed{= \quad 1}$$

How does a change in weight w_1 impact output?



Continue the **chain rule**

$$\frac{d\ out}{d\ w_1} = \frac{d\ out}{d\ s_2} \frac{d\ s_2}{d\ x_2} \frac{d\ x_2}{d\ w_1}$$

wiggle x_2 and s_2 changes

$$s_2 = S(x_2) = \left(\frac{1}{1+e^{-x_2}}\right)$$

so $$\frac{d\ s_2}{d\ x_2} = \frac{d}{dx_2}\left(\frac{1}{1+e^{-x_2}}\right)$$

A complicated derivative

We need to compute:

$$\frac{d}{dx_2}\left(\frac{1}{1+e^{-x_2}}\right) = \text{Hmm } \ldots \text{ } \textbf{Quotient Rule}$$

$$= \frac{(1+e^{-x_2})\frac{d(1)}{dx_2} - (1)\frac{d(1+e^{-x_2})}{dx_2}}{(1+e^{-x_2})^2}$$

0

Q: What is $\dfrac{d(1+e^{-x_2})}{dx_2}$?

= - 1

$$= \frac{d(1)}{dx_2} + \frac{d(e^{-x_2})}{dx_2} = (e^{-x_2})\frac{d(-x_2)}{dx_2} = -e^{-x_2}$$

0

A complicated derivative

So,

$$\frac{d}{dx_2}\Big(\frac{1}{1+e^{-x_2}}\Big) = \frac{0-(1)(-e^{-x_2})}{(1+e^{-x_2})^2}$$

$$= \frac{e^{-x_2}}{(1+e^{-x_2})^2}$$

Q: Can we simplify this expression?

A: Yes, but requires faith …

- Split up denominator: $\frac{e^{-x_2}}{(1+e^{-x_2})^2} = \Big(\frac{1}{1+e^{-x_2}}\Big)\Big(\frac{e^{-x_2}}{1+e^{-x_2}}\Big)$

A complicated derivative

- Add 0 to the second term:

Sum is 0

$$\left(\frac{1}{1+e^{-x_2}}\right)\left(\frac{e^{-x_2}}{1+e^{-x_2}}\right) = \left(\frac{1}{1+e^{-x_2}}\right)\left(\frac{1+e^{-x_2}-1}{1+e^{-x_2}}\right)$$

Q: Why? **A:** Let's organize terms …

$$= \left(\frac{1}{1+e^{-x_2}}\right)\left(\frac{1+e^{-x_2}}{1+e^{-x_2}} - \frac{1}{1+e^{-x_2}}\right)$$

$= s_2$ $= 1$ $= s_2$

Remember:

$$s_2 = \left(\frac{1}{1+e^{-x_2}}\right)$$

$= (s_2)\ (1 - s_2)$

A complicated derivative

So, for our chain rule calculation:

$$\frac{d\,out}{d\,w_1} = \frac{d\,out}{d\,s_2} \boxed{\frac{d\,s_2}{d\,x_2}} \frac{d\,x_2}{d\,w_1}$$

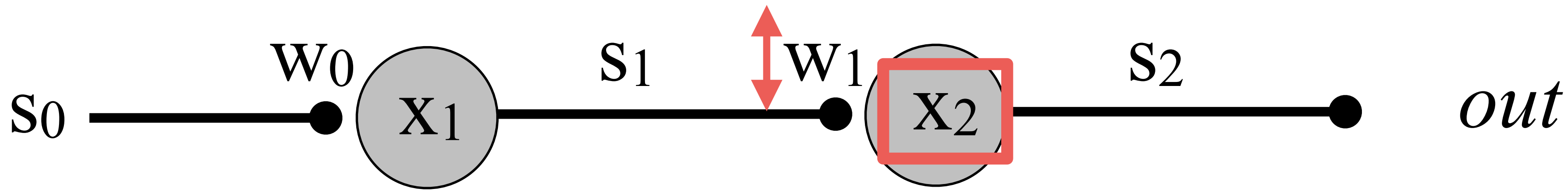
And we found:

$$\frac{d\,s_2}{d\,x_2} = \frac{d}{dx_2}\left(\boxed{\frac{1}{1+e^{-x_2}}}\right) = \ldots \text{ many steps } \ldots \boxed{= s_2\,(1 - s_2)}$$

$= s_2$

To complete the chain rule, one more derivative …

How does a change in weight w_1 impact output?



Continue the **chain rule**:

$$\frac{d\,out}{d\,w_1} = \frac{d\,out}{d\,s_2}\;\frac{d\,s_2}{d\,x_2}\;\boxed{\frac{d\,x_2}{d\,w_1}}$$

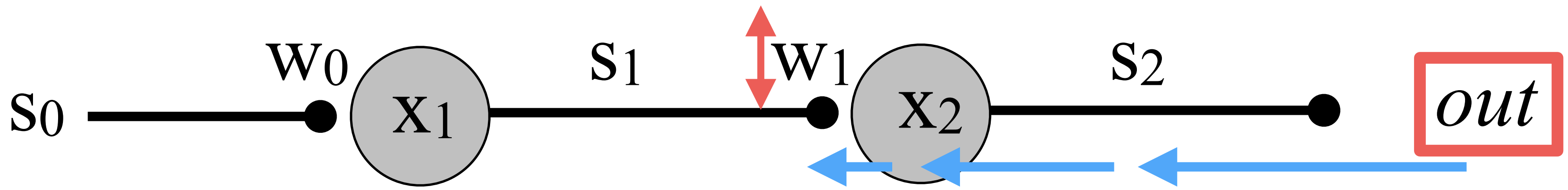
wiggle w_1 and x_2 changes

Remember: $x_2 = s_1 w_1$

$$\frac{d\,x_2}{d\,w_1} = \frac{d\,(s_1 w_1)}{d\,w_1} = \boxed{s_1}$$

How does a change in weight w_1 impact output?

Back to our original question:



Q: How does a change in weight w_1 impact the output?

Mathematically … the **chain rule**

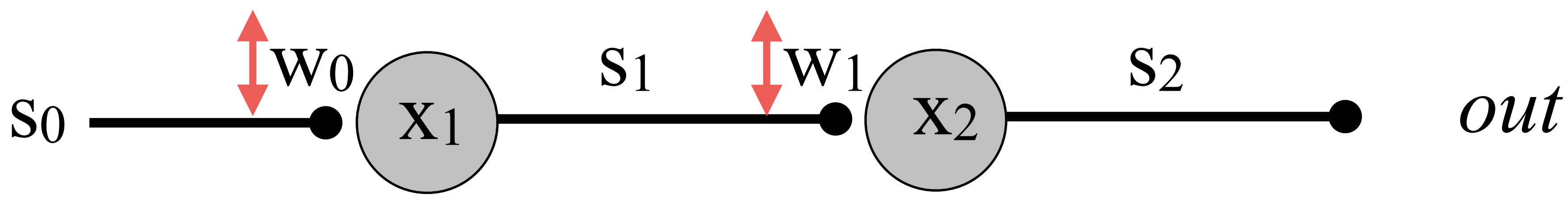
$$\frac{d\,out}{d\,w_1} = \frac{d\,out}{d\,s_2} \quad \frac{d\,s_2}{d\,x_2} \quad \frac{d\,x_2}{d\,w_1}$$

$$\frac{d\,out}{d\,w_1} = 1 \quad s_2\,(1 - s_2) \quad s_1$$

Slide 16 *Slide 21* *Slide 22*

How does a change in weight w_0 impact output?

We're almost there …



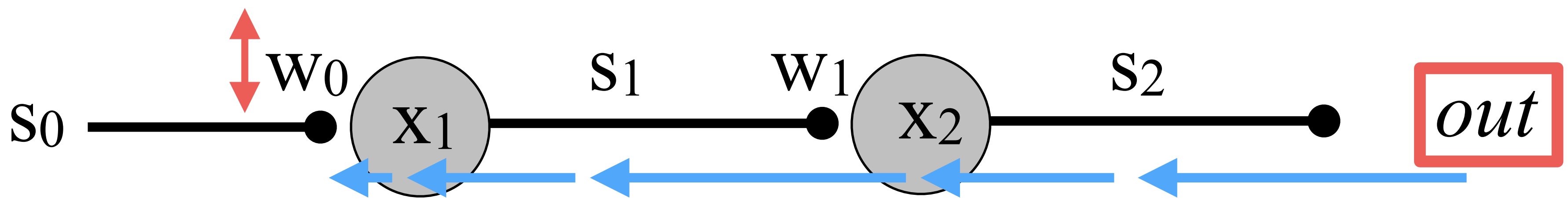
$$\frac{d\,out}{d\,w_1} = s_2\,(1 - s_2)\; s_1$$

Q: How does a change in weight w_0 impact the output?

A: Chain rule …

How does a change in weight w_0 impact output?

Q: How does a change in weight w_0 impact the output?



$$\frac{d\,out}{d\,w_0} = \frac{d\,out}{d\,s_2} \quad \frac{d\,s_2}{d\,x_2} \quad \boxed{\frac{d\,x_2}{d\,s_1} \quad \frac{d\,s_1}{d\,x_1} \quad \frac{d\,x_1}{d\,w_0}}$$

Ugh …

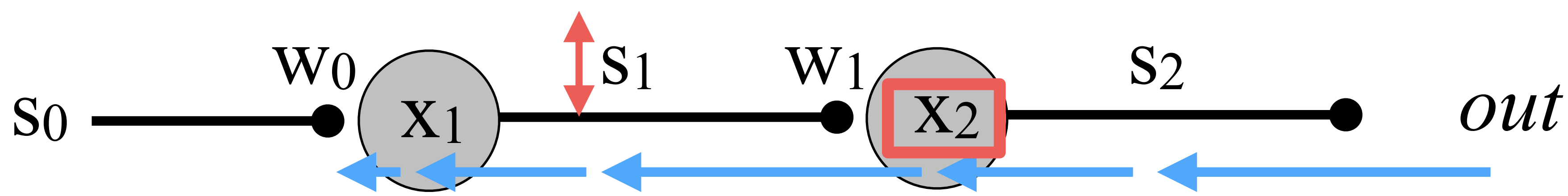
We've already calculated two of these.

w_2 $\quad$ $s_2\,(1 - s_2)$

Let's compute the last 3 terms …

How does a change in weight w_0 impact output?

3rd term:



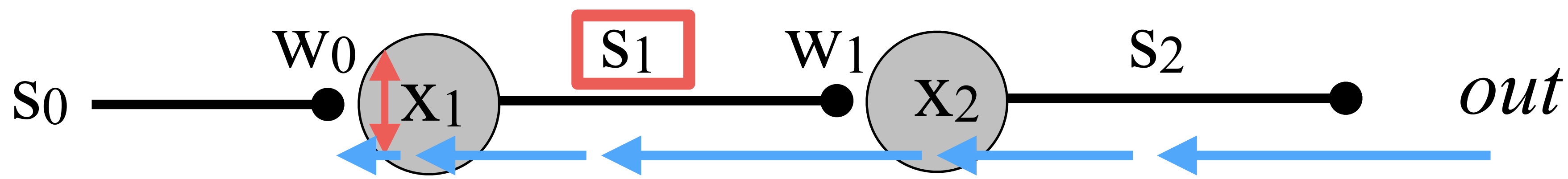
$$\frac{d\,out}{d\,w_0} = \frac{d\,out}{d\,s_2}\;\frac{d\,s_2}{d\,x_2}\;\boxed{\frac{d\,x_2}{d\,s_1}}\;\frac{d\,s_1}{d\,x_1}\;\frac{d\,x_1}{d\,w_0}$$

Remember: $x_2 = s_1 w_1$

$$\frac{d\,x_2}{d\,s_1} = \frac{d\,(\,s_1 w_1\,)}{d\,s_1} = \boxed{w_1}$$

How does a change in weight w_0 impact output?

4th term:



$$\frac{d\,out}{d\,w_0} = \frac{d\,out}{d\,s_2}\;\frac{d\,s_2}{d\,x_2}\;\frac{d\,x_2}{d\,s_1}\;\boxed{\frac{d\,s_1}{d\,x_1}}\;\frac{d\,x_1}{d\,w_0}$$

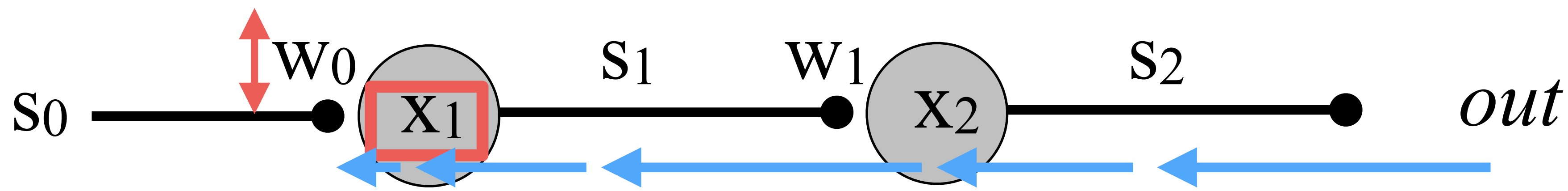
We found earlier that:

$$\frac{d\,s_2}{d\,x_2} = s_2\,(1 - s_2) \qquad \ldots \text{so} \ldots \qquad \frac{d\,s_1}{d\,x_1} = \boxed{s_1\,(1 - s_1)}$$

This involved many steps

How does a change in weight w_0 impact output?

5th term:



$$\frac{d\,out}{d\,w_0} = \frac{d\,out}{d\,s_2} \frac{d\,s_2}{d\,x_2} \frac{d\,x_2}{d\,s_1} \frac{d\,s_1}{d\,x_1} \boxed{\frac{d\,x_1}{d\,w_0}}$$

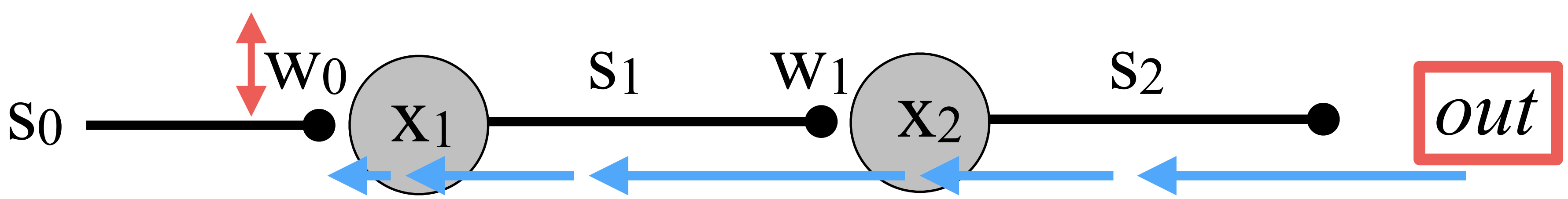
Remember: $x_1 = s_0 w_0$

$$\frac{d\,x_1}{d\,w_0} = \frac{d\,(s_0 w_0)}{d\,w_0} = \boxed{s_0}$$

How does a change in weight w_0 impact output?

We now have the pieces to answer:

Q: How does a change in weight w_0 impact the output?



$$\frac{d\ out}{d\ w_0} = \frac{d\ out}{d\ s_2} \quad \frac{d\ s_2}{d\ x_2} \quad \frac{d\ x_2}{d\ s_1} \quad \frac{d\ s_1}{d\ x_1} \quad \frac{d\ x_1}{d\ w_0}$$

$= \ 1$ — *Slide 16*

$s_2(1 - s_2)$ — *Slide 21*

w_1 — *Slide 26*

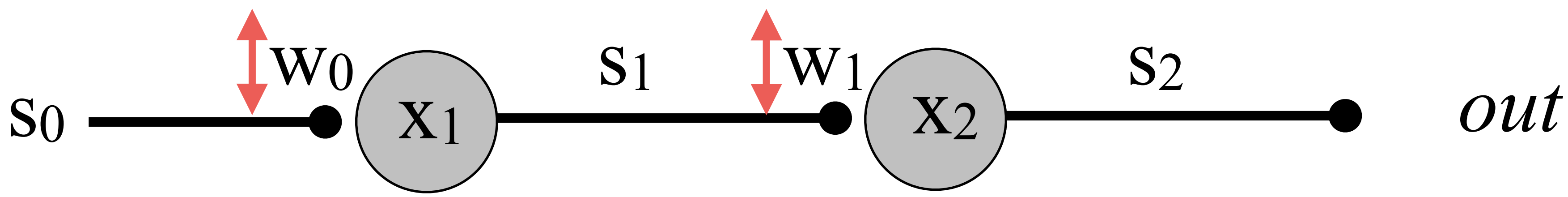
$s_1(1 - s_1)$ — *Slide 27*

s_0 — *Slide 28*

How does a change in weight impact output?

To summarize:

– We've found how changes in model weights impact output.

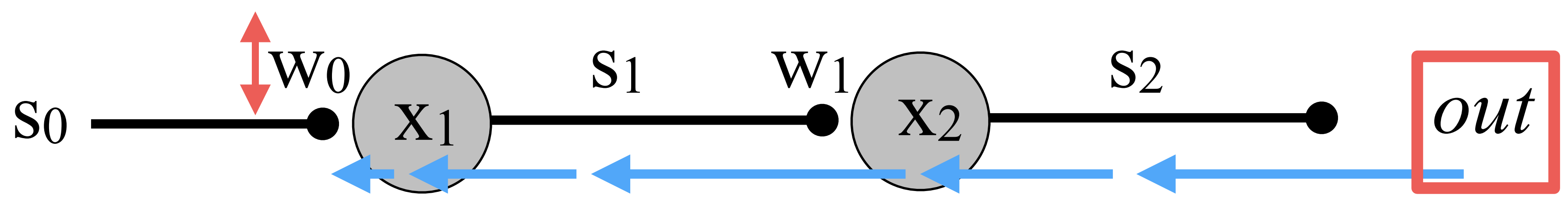


$$\frac{d\,out}{d\,w_1} = s_2 (1 - s_2)\ s_1$$

$$\frac{d\,out}{d\,w_0} = s_2 (1 - s_2)\ w_1\ s_1 (1 - s_1)\ s_0$$

So, how does a change in weight impact output? **backpropagation**!

How does a change in weight impact output?



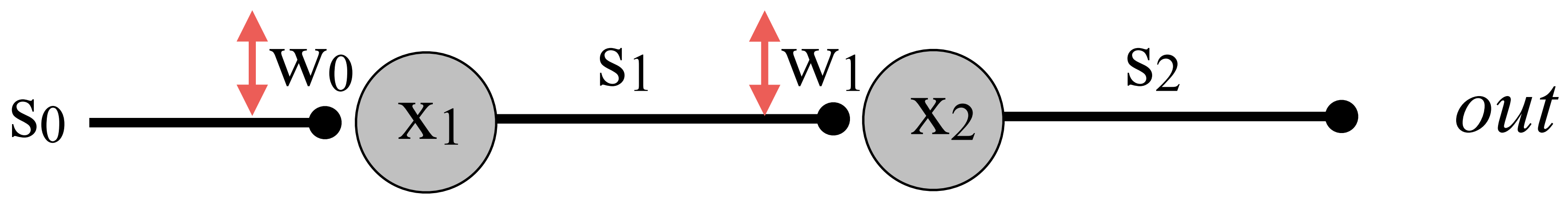
Backpropagation:

Work "backwards" from output to weight, computing derivatives along the way

Q: How do these derivatives help us update weights and obtain desired output?

Define our goal

We want:



$$out = \text{target}$$

rearrange $\quad out - \text{target} = 0 \quad$ **GOAL**

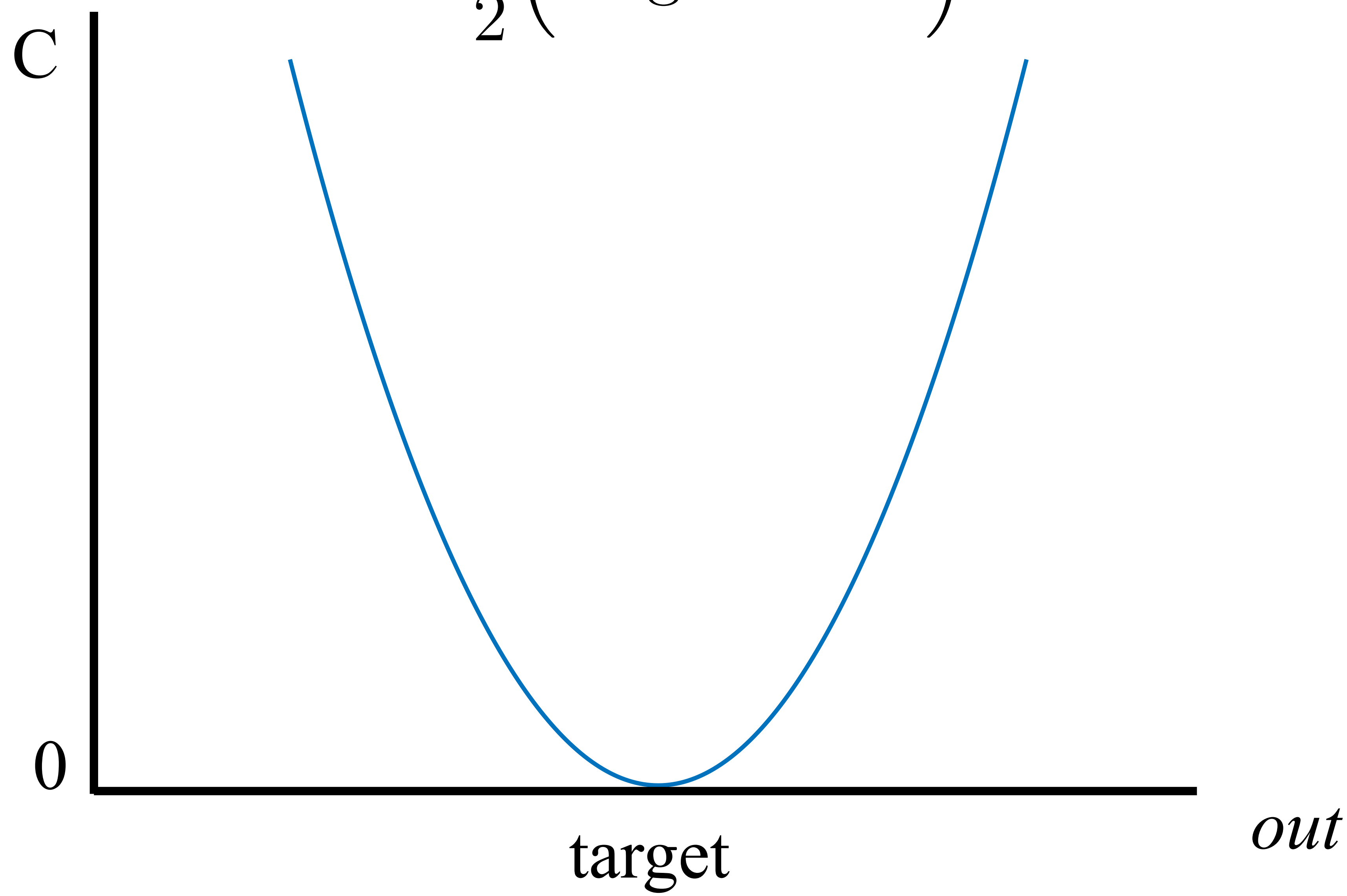
Let's use this to define a **cost function** …

Create a cost function

Define the cost function:

$$C = \frac{1}{2}\left(\text{target} - out\right)^2$$

Plot it:



Q: Where is the cost zero? **A:** When *out* = target.

Create a cost function

Q: Why this cost function?

$$C = \frac{1}{2}\Big(\text{target} - out\Big)^2$$

- Minimum (the "lowest cost") when *out* = target.
- It's convenient (a quadratic).
- It steadily increases as *out* deviates from target.
- It's "easy" to compute derivatives.

Create a cost function

Q: How does the cost function <u>change</u> due to changes in *out*?

A: We need to compute a derivative …

$$\frac{dC}{dout} = \frac{d}{out}\left[\frac{1}{2}\left(\text{target} - out\right)^2\right] = ?$$

Chain rule …

$$\frac{dC}{dout} = 2\frac{1}{2}\left(\text{target} - out\right)^1\left(\frac{d\left(-out\right)}{d\,out}\right)$$

1   -1

$$\frac{dC}{dout} = -\left(\text{target} - out\right)$$

$$\boxed{\frac{dC}{dout} = out - \text{target}}$$

Create a cost function

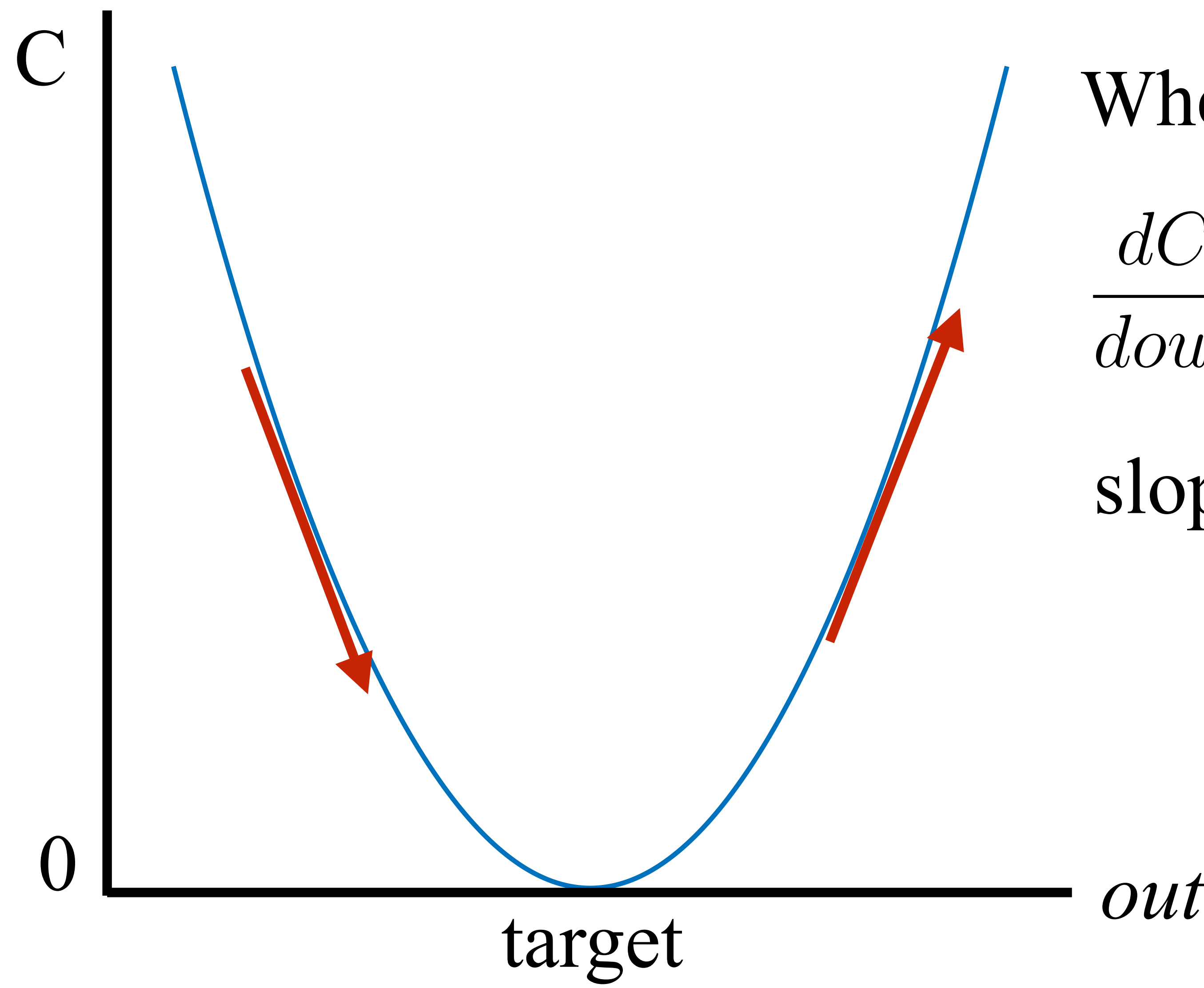
Q: Does this derivative make sense?

$$\frac{dC}{dout} = out - \text{target}$$

When *out* < target

$$\frac{dC}{dout} < 0$$

slope < 0



When *out* > target

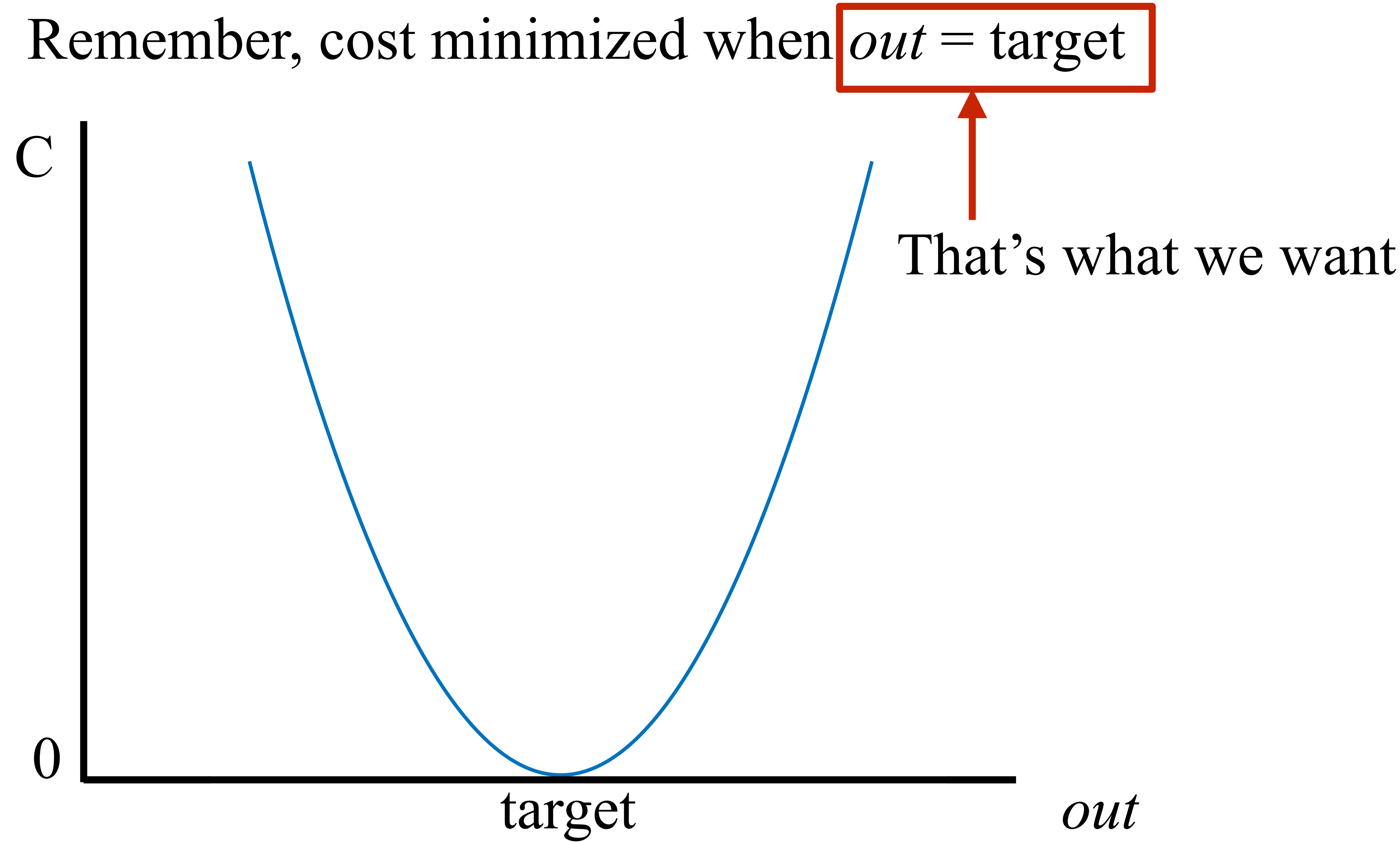
$$\frac{dC}{dout} > 0$$

slope > 0

Create a cost function

Now, our goal: Choose weights to minimize the cost function

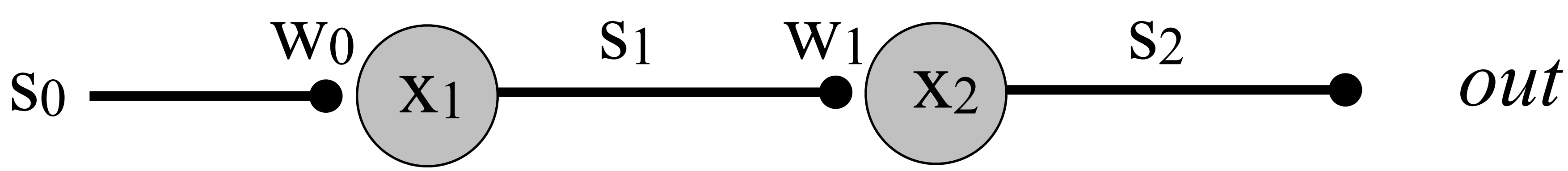
Remember, cost minimized when *out* = target



Here, we plot C versus *out*. But *out* depends on weights …

Create a cost function

Q: How does *out* depend on weights?



A: It's complicated.

Consider feedforward solution …

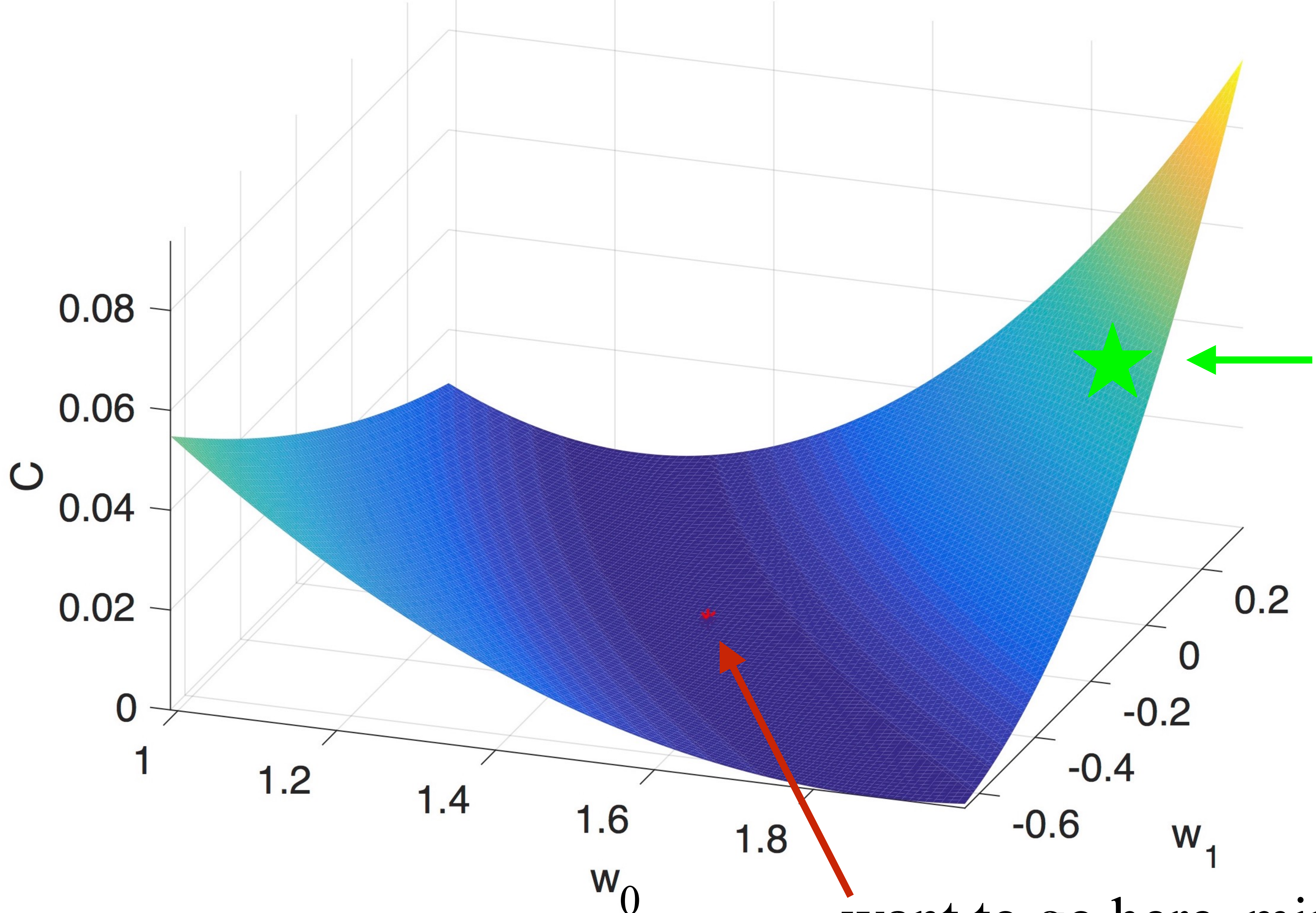
$$x_1 = s_0\, w_0 \rightarrow s_1 = S(x_1) \rightarrow x_2 = s_1\, w_1 \rightarrow s_2 = S(x_2) \rightarrow \boxed{out} = s_2$$

If *out* depends on weights, so does the cost … $C(w_0, w_1) = ?$

Create a cost function

Plot cost C versus w_0 and w_1:

<u>Goal</u>: minimize cost.



But, we have our initial, randomly chosen weights

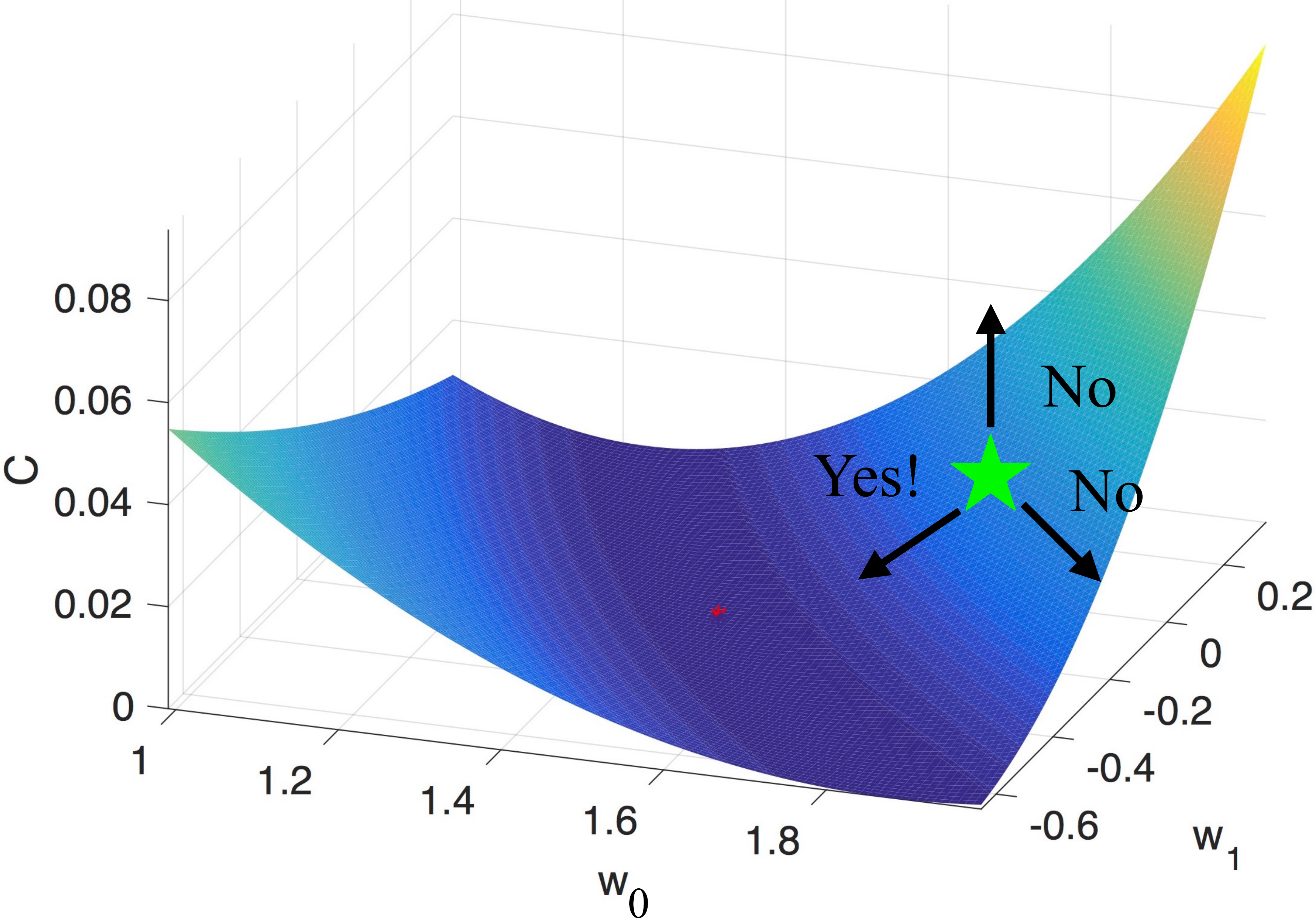
want to go here, minimize cost.

Q: How do we adjust weights to reach the minimum cost?

A: Move "downhill" …

Follow the cost function

Intuition: move down the **steepest direction** of cost function



Imagine placing a marble … where does it roll? To the minimum.

Q: How do we find the steepest direction? **A:** Compute the gradient

Follow the cost function

Gradient of the cost function.

How C changes due to small changes in w_0, w_1.

We need to compute: $$\frac{dC}{dw_0} \qquad \frac{dC}{dw_1}$$

Then, update the weights in steps proportional to the negative gradient.

$$w_0 \leftarrow w_0 - \alpha \frac{dC}{dw_0} \qquad w_1 \leftarrow w_1 \boxed{- \alpha \frac{dC}{dw_1}}$$

update

"new" weight

"original" weight

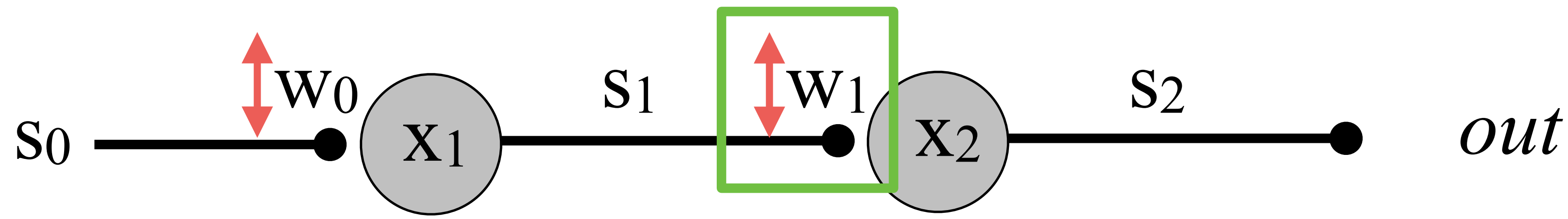
(negative) steepest direction of C

Procedure: gradient descent

α = learning rate

Follow the cost function

Q: How does the cost function change due to change in w_1?



$$\frac{d\,C}{d\,w_1} = \quad ???$$

We don't know this … but can write it using things we do know.

We know how C depends on *out*: $\mathrm{C} = \frac{1}{2}\Big(\text{target} - out\Big)^2$

And we know how *out* depends on w_1

To compute the derivative, use the **chain rule** …

Follow the cost function

Our goal:

$$\frac{d\,C}{d\,w_1} = \;???$$

We know C depends on *out*, and *out* depends on w_1 …

$$\frac{d\,C}{d\,w_1} = \frac{d\,C}{d\,out}\boxed{\frac{d\,out}{d\,w_1}}$$

We've already solved the first derivative!

$$\frac{dC}{dout} = out - \text{target}$$

Slide 35

Let's compute the next derivative …

Follow the cost function

$$\frac{d\,out}{d\,w_1} = ???$$

We've already done this …

$$\frac{\mathrm{d}\,out}{\mathrm{d}\,w_1} = s_2\,(1 - s_2)\,s_1$$

(*Slide 23*)

Then,

$$\frac{d\,C}{d\,w_1} = \frac{d\,C}{d\,out}\;\frac{d\,out}{d\,w_1}$$

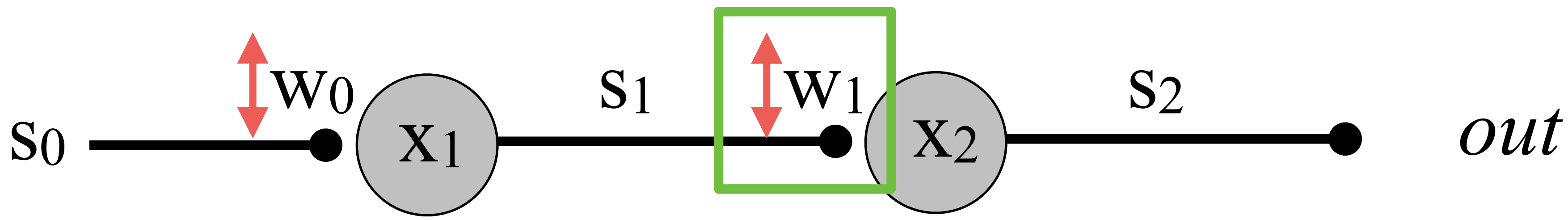
$$\frac{d\,C}{d\,w_1} = \boxed{(out - \text{target})}\; s_2\,(1 - s_2)\,s_1$$

How bad we're doing.

complicated expression of outputs

Follow the cost function

Q: How does the cost function change due to change in w_1?



Update the weight w_1:

$$w_1 \leftarrow w_1 - \alpha \frac{dC}{dw_1}$$

substitute in for this

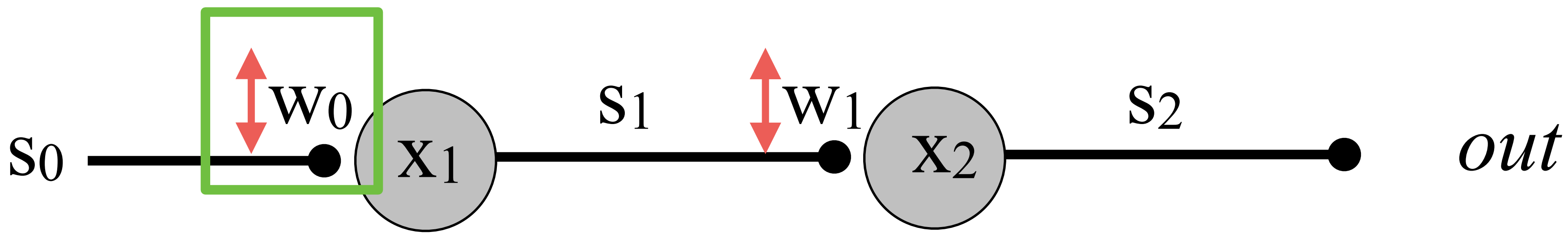
becomes

$$w_1 \leftarrow w_1 - \alpha (out - \text{target}) \quad s_2(1 - s_2)s_1$$

Q: What happens when *out* = target?

Follow the cost function

Q: How does the cost function change due to change in w_0?

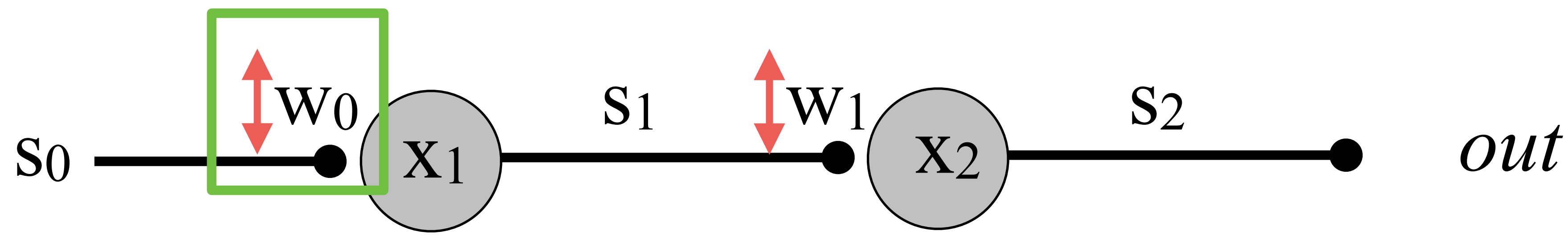


$$\frac{d\,C}{d\,w_0} = \quad ???$$

Try it …

Follow the cost function

Q: How does the cost function change due to change in w_0?



We conclude:

$$\frac{dC}{dw_0} = (out - \text{target})\ s_2(1 - s_2)\ w_1\ s_1(1 - s_1)\ s_0$$

and

$$w_0 \leftarrow w_0 - \alpha(out - \text{target}) \quad s_2(1 - s_2)w_1 s_1(1 - s_1)s_0$$

Impressive expression

Put it all together

Prescription to find the weights that minimize cost function (so that *out* is near target).

1. Choose random initial weights.

$w_0 = 2 \qquad w_1 = 1$

2. Fix input at desired value, and calculate *out*.

s_0 —— **forward propagation** ——→ *out*

Put it all together

Prescription (continued)

3. Update the weights

$$w_1 \leftarrow w_1 - \alpha(out - \text{target}) \quad s_2(1 - s_2)s_1$$

$$w_0 \leftarrow w_0 - \alpha(out - \text{target}) \quad s_2(1 - s_2)w_1 s_1(1 - s_1)s_0$$

Note: We know all of the values required

α = learning rate, we choose this.

s_0, s_1, s_2 = calculated during forward propagation

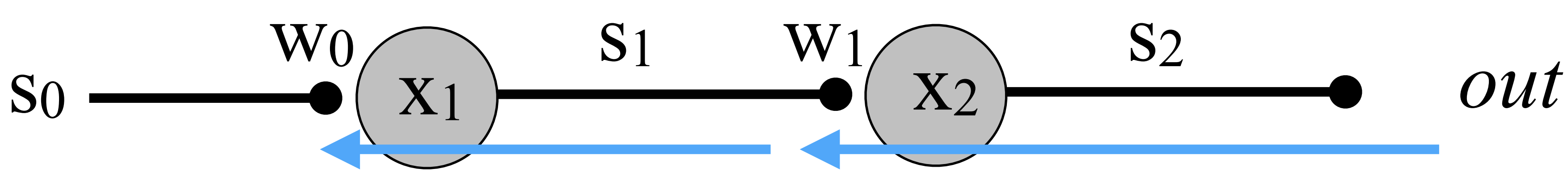
Put it all together

Prescription (continued)

4. Repeat Steps 2 & 3 until error is small enough.

or *out* is close enough to target.

This procedure is called **backpropagation**

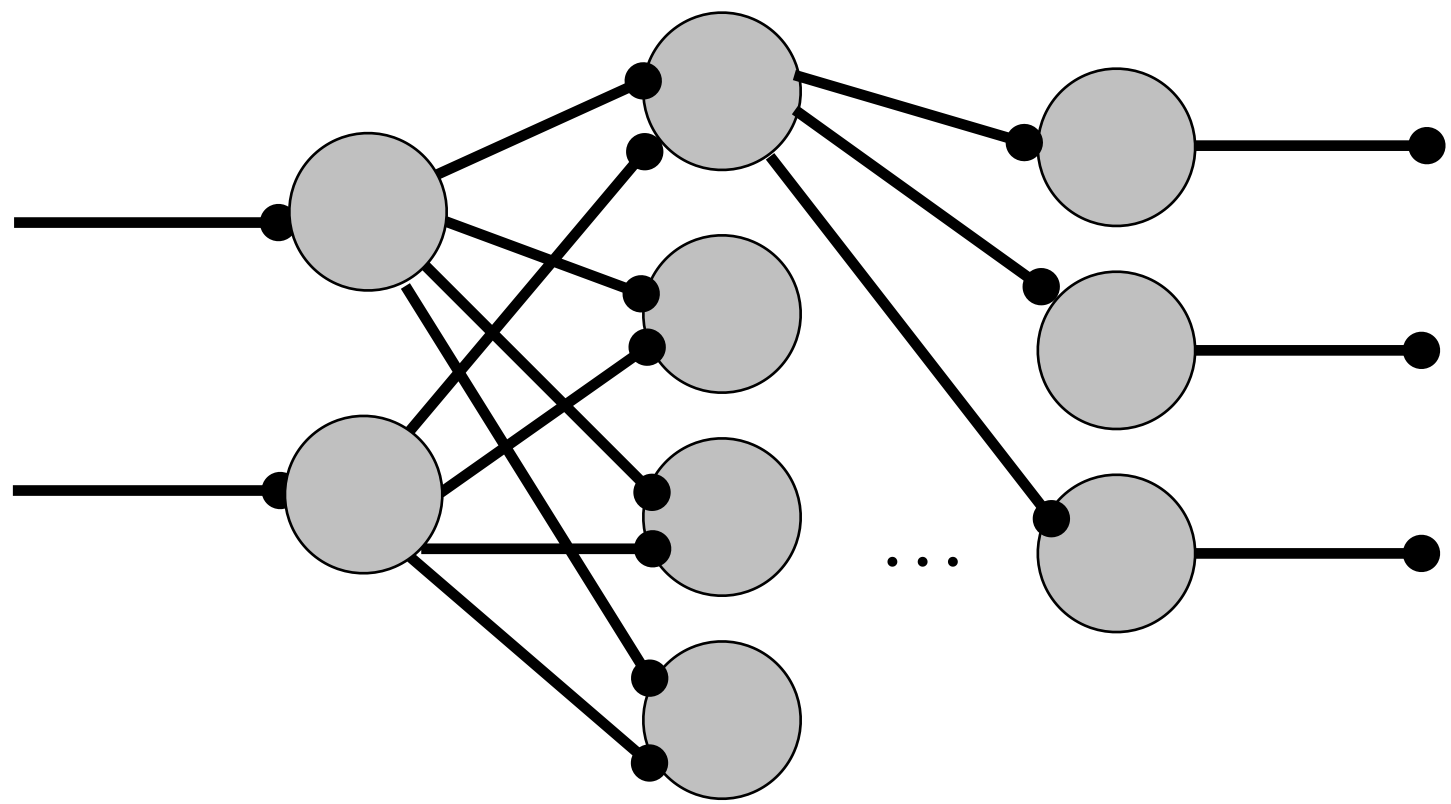


we work "backwards" through our neural network

from *out* to changes in w_1 to changes in w_0

Backpropagation

Can evaluate more complicated neural networks



Same ideas apply, but algebra is intense.

Example: playground.tensorflow.org

Next …

Implement backpropagation in Python

s_0	*out*

```
[[ 0.32625098  0.71507894]
 [-0.65595776  1.0065779 ]
 [-0.9664146   1.09676408]
 ...
 [ 1.53148642  0.48558244]
 [-1.07312405  1.12584863]
 [ 1.32680081  0.51212226]]
```

Q: What weights (w_0 w_1) produced these data?