Linear algebra

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(borrowed from *Math Tools for Neuroscience*, Prof. Jonathon Pillow, Princeton University)

Linear algebra

"Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier."

- Glibert Strang, *Linear algebra and its applications*

Topics

- vectors
  - addition
  - scalar multiplication
- vector norm
- unit vectors
- dot product
- linear projection
- **orthogonality**

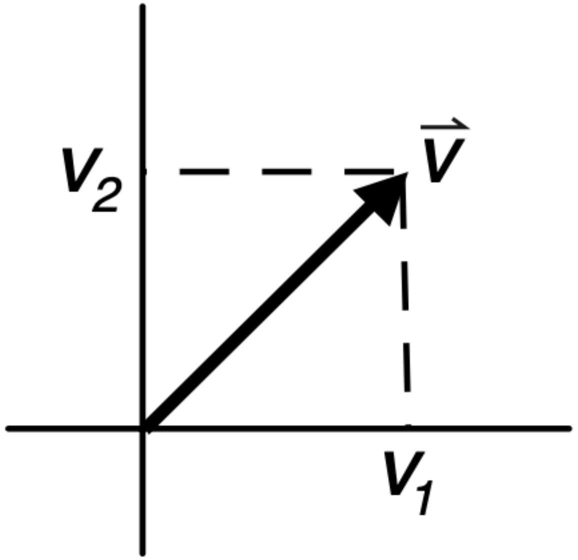
Vectors

a **binary word** [0,0,1,1,1,1,0,1, …]

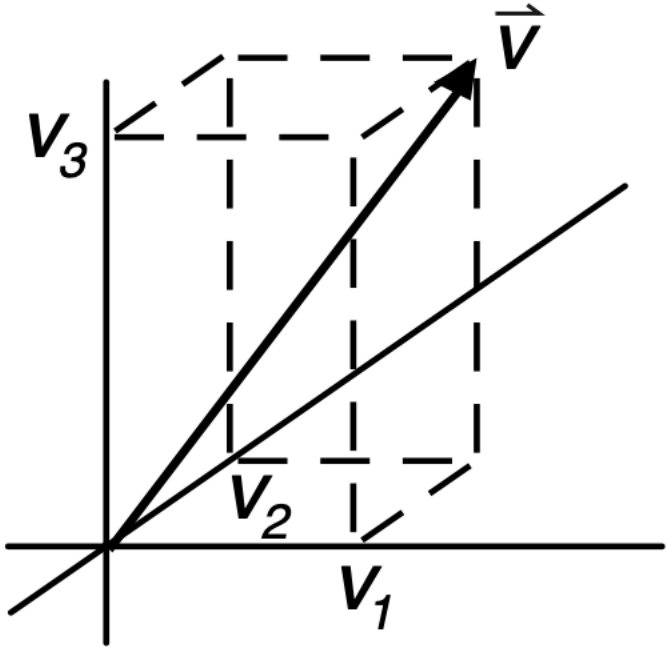
$$\xi = [1,1,-1,1,1,-1,1,1,1,1]$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

(N numbers)



(just 2 numbers)



(just 3 numbers)

column vector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

```
# make a 3-component column vector
v = np.array([[3], [1], [-7]])
```

transpose

$$\vec{v}^T = \begin{pmatrix} v_1 & v_2 & \cdots & v_N \end{pmatrix}$$

row vector

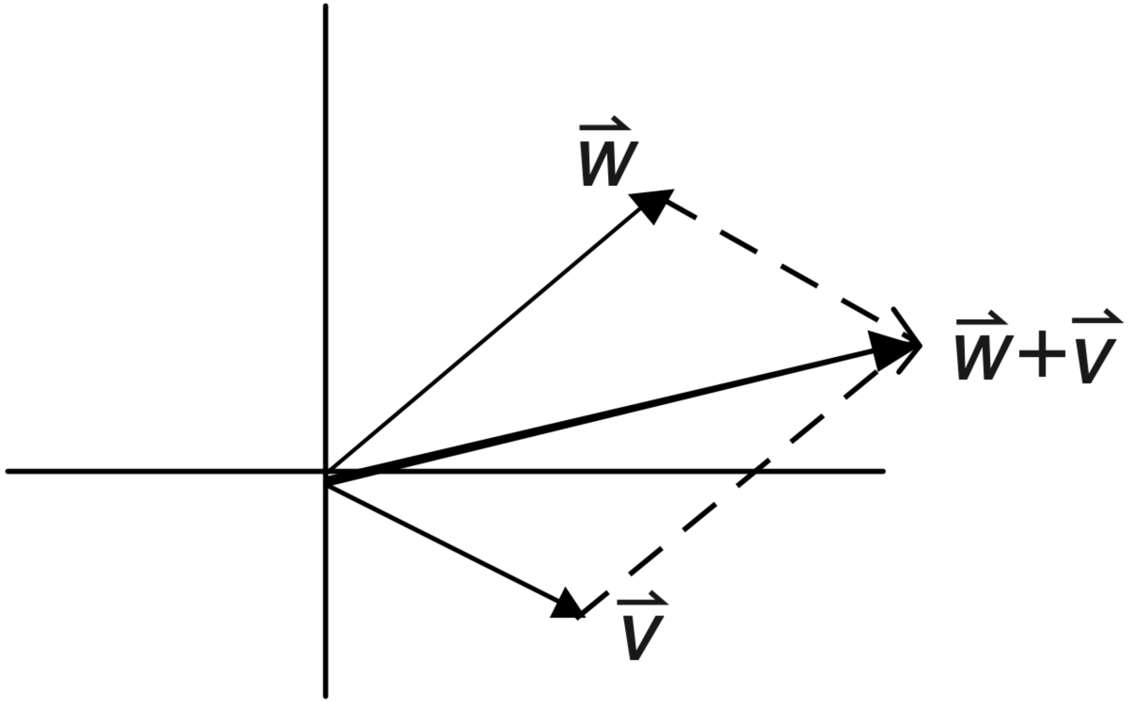
```
# transpose
v.T
```

```
# create row vector directly
v = np.array([[3,1,-7]]) # row vector
# or
v = np.array([3,1,-7]) # 1D vector
```

Addition of vectors

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

Consider N = 2

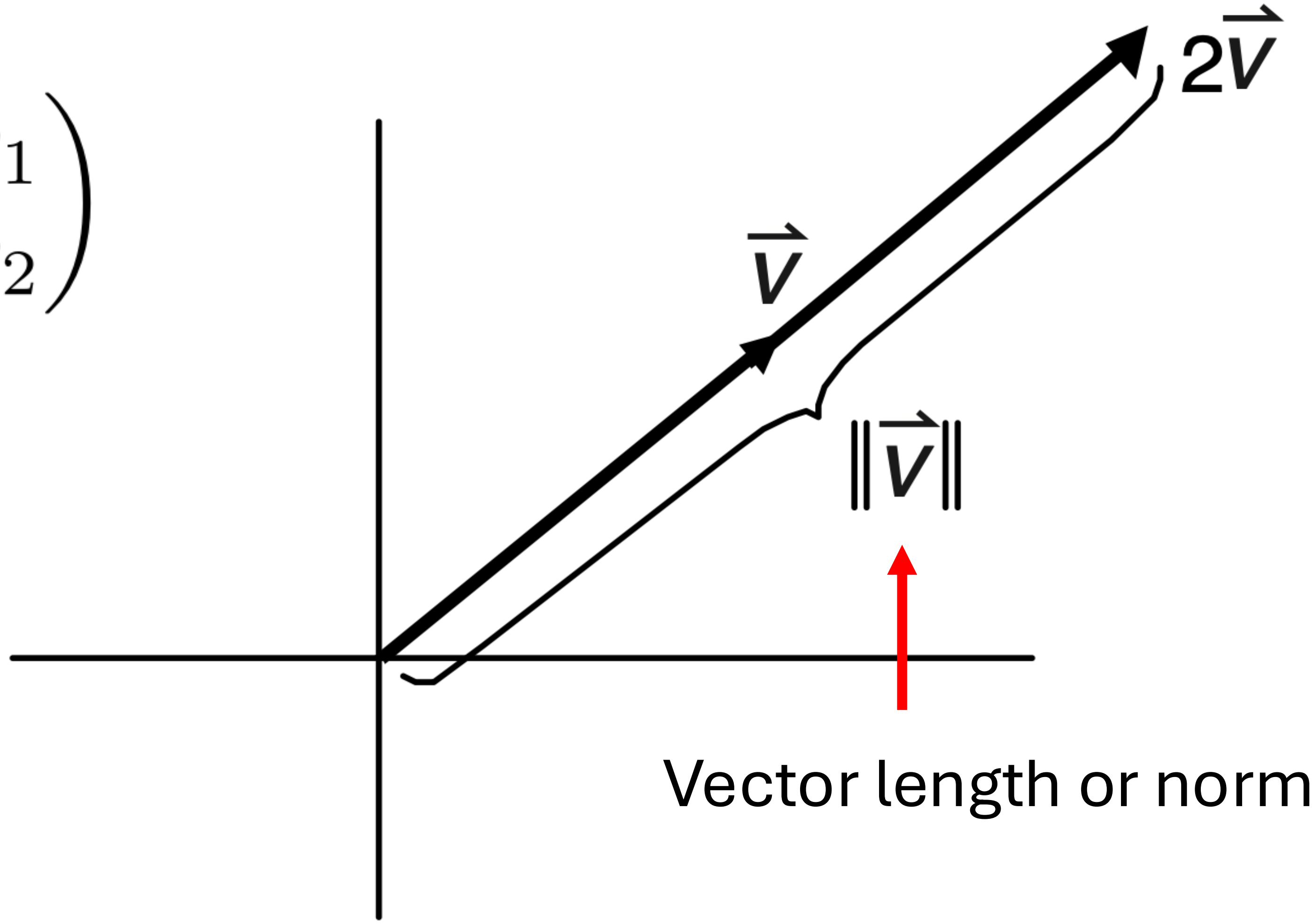


Scalar multiplication

$$a\vec{v} = a\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 \\ av_2 \end{pmatrix}$$

Consider N = 2

Example a = 2



Vector norm ("L2 norm")

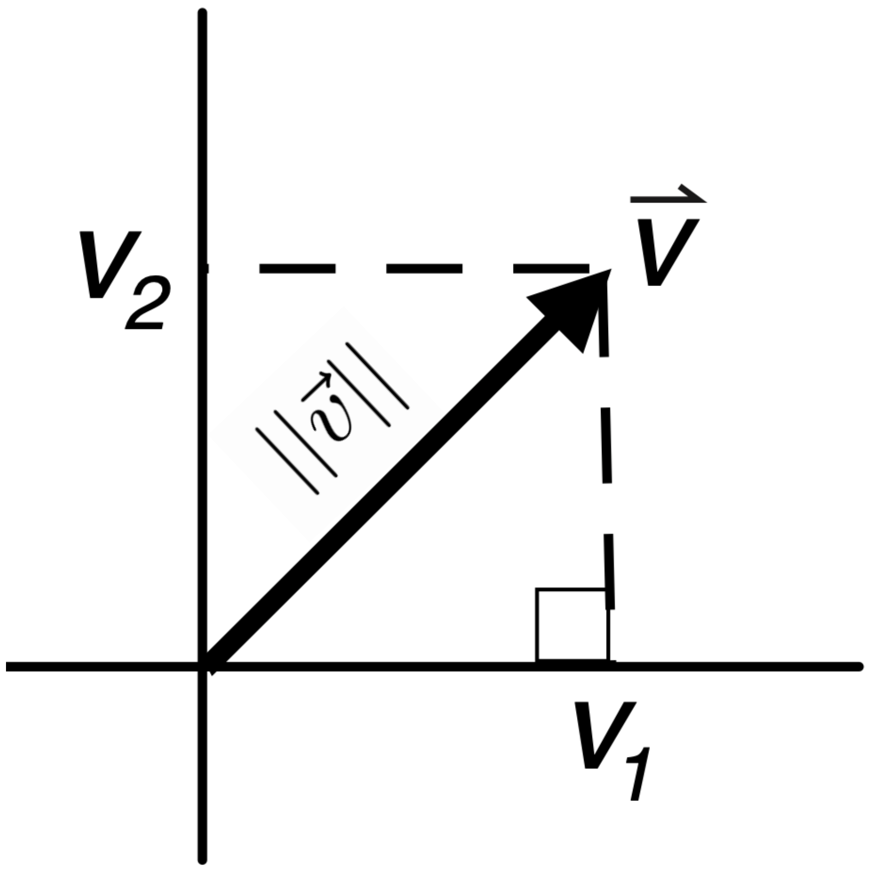
- vector length in Euclidean space

<u>In 2-D:</u>

$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2}$$

<u>In *n*-D:</u>

$$||\vec{v}|| = \sqrt{v_1^2 + \cdots + v_n{}^2}$$



Vector norm (“L2 norm”)

Exercises compute the vector length (norm of each of the following vectors):

a) [7, 7]
b) [5,5,5,5]
c) [10, 1]
d) [5,5,7]

Vector norm ("L2 norm") in Python

```
# make a vector
v = np.array([1, 7, 3, 0, 1])

# many equivalent ways to compute norm
np.linalg.norm(v)       # built-in function
np.sqrt(np.dot(v,v))    # sqrt of dot product
np.sqrt(v.T @ v)        # sqrt of v-tranpose times v
np.sqrt(sum(v * v))     # sqrt of sum of elementwise product
np.sqrt(sum(v ** 2))    # sqrt of v elementwise-squared

# note use of @ and * and **
# @ - gives matrix multiply
# * - gives elementwise multiply
# ** - gives exponentiation ("raising to a power")
```

Unit vector

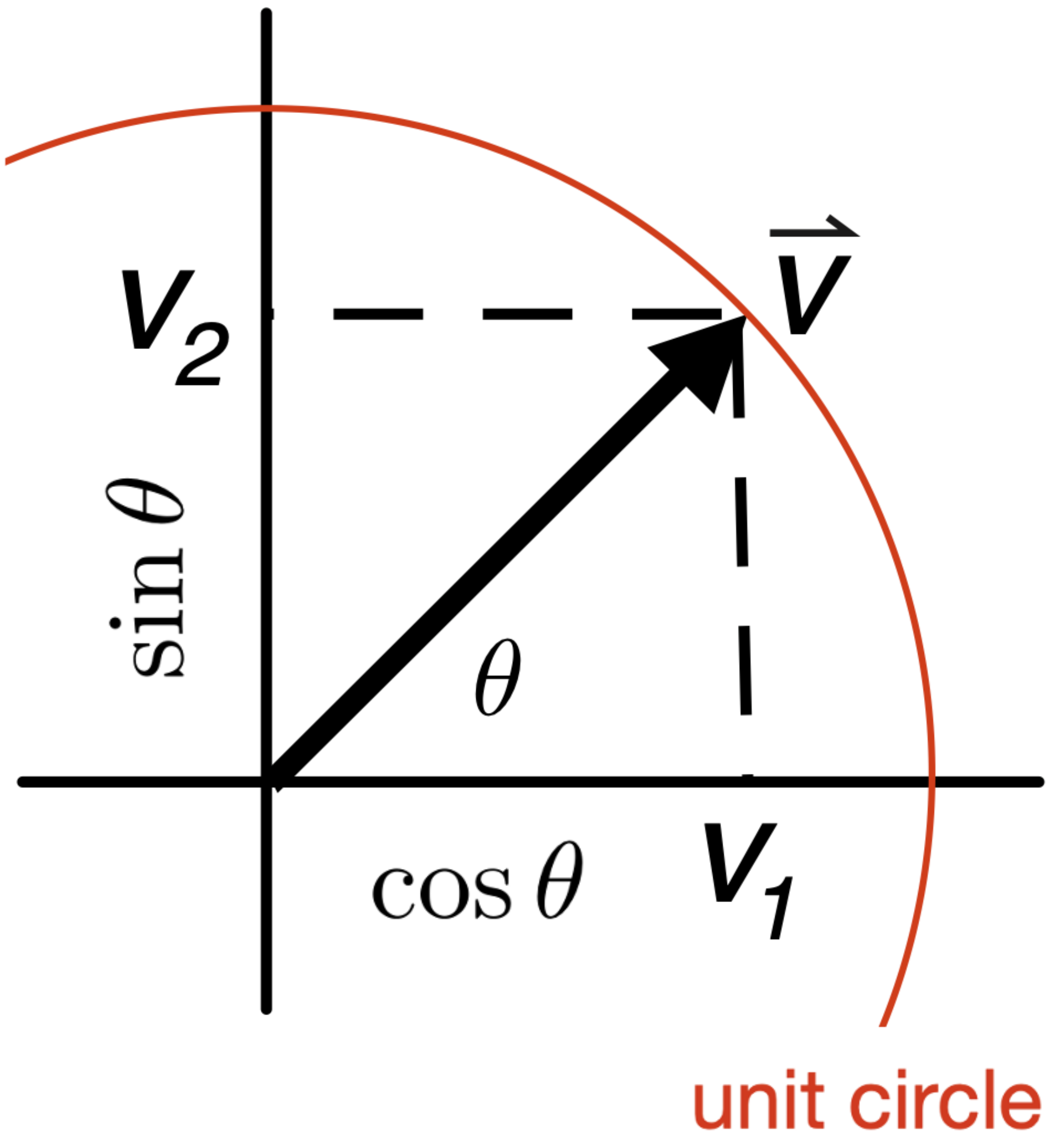
vector such that $||\vec{v}|| = 1$

<u>in 2 dimensions</u>

$$\vec{v} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

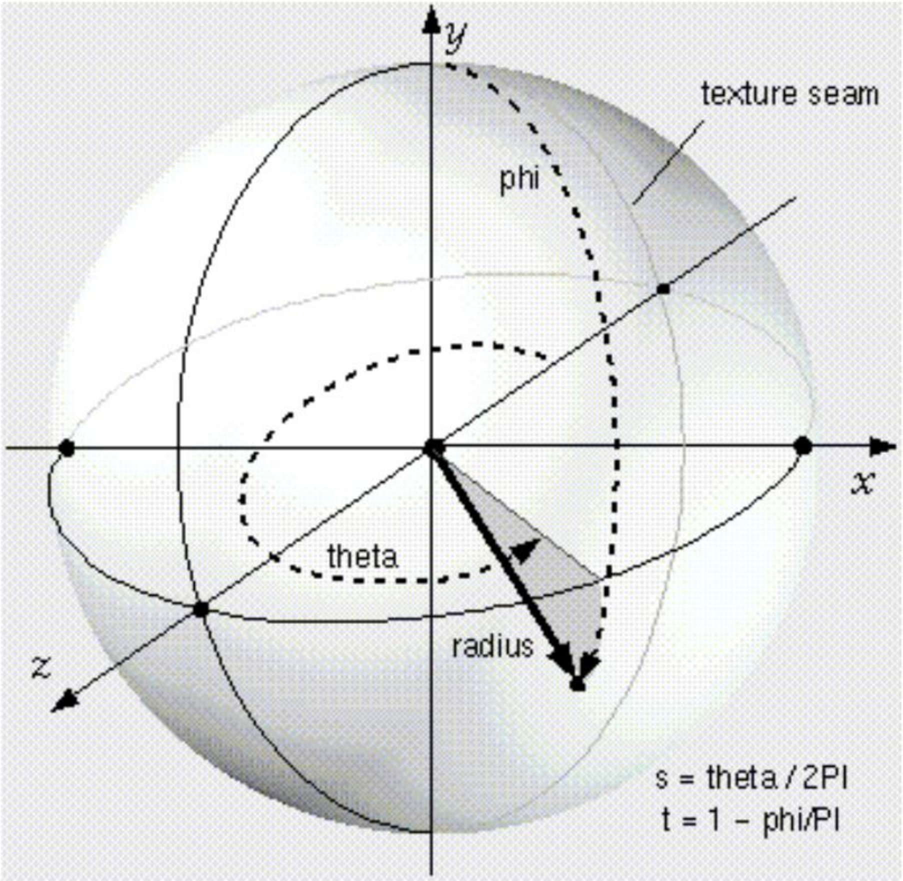
$$\cos^2\theta + \sin^2\theta = 1$$

START HERE



Unit vector

vector such that $||\vec{v}|| = 1$



in *n* dimensions

$$v_1^2 + v_2^2 + \ldots + v_n^2 = 1$$

sits on the surface of an *n*-dimensional hypersphere

Unit vector

vector such that $||\vec{v}|| = 1$

We can make any vector into a unit vector

$$\vec{v} \rightarrow \frac{1}{||\vec{v}||}\vec{v}$$

Inner product (aka "dot product")

- produces a scalar from two vectors

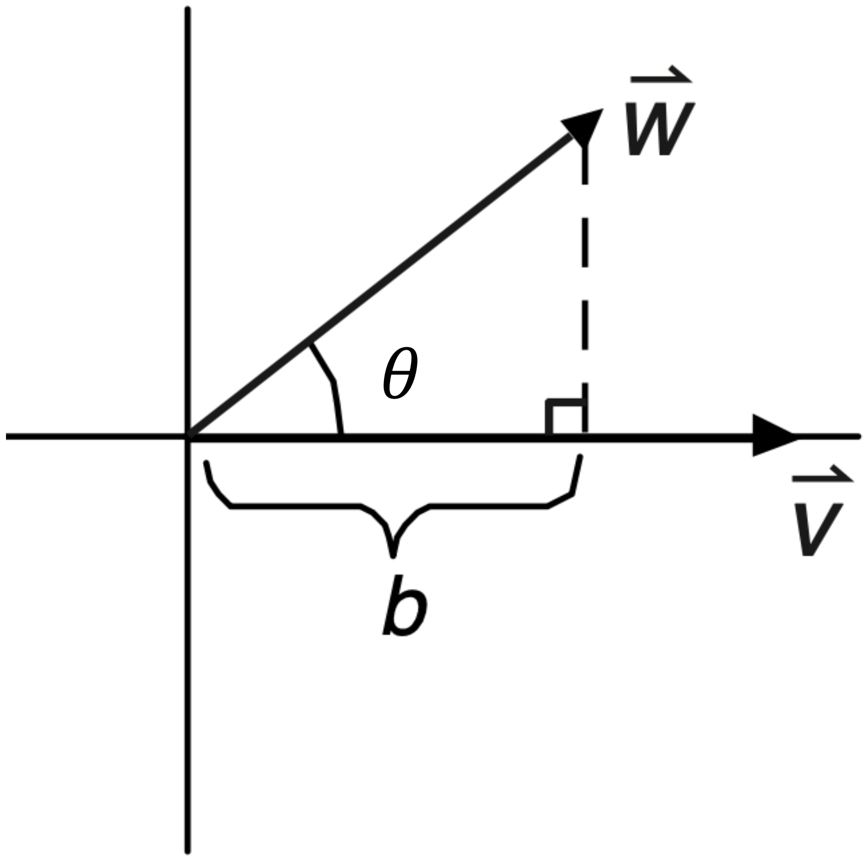
$$\vec{v} \cdot \vec{w}$$

$$\langle \vec{v}, \vec{w} \rangle$$

$$v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

$$\|v\| \|w\| \cos\theta$$

$$\vec{v}^T \vec{w} = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$



Inner product (aka “dot product”)

Exercises:

v1 = [1,2,3]
v2 = [3,2,-1]
v3 = [10,0,5]

Compute:
v1•v2, v1•v3, v2•v3

Linear projection

Intuitively, drop a vector down onto a line (or linear surface) at a right angle.

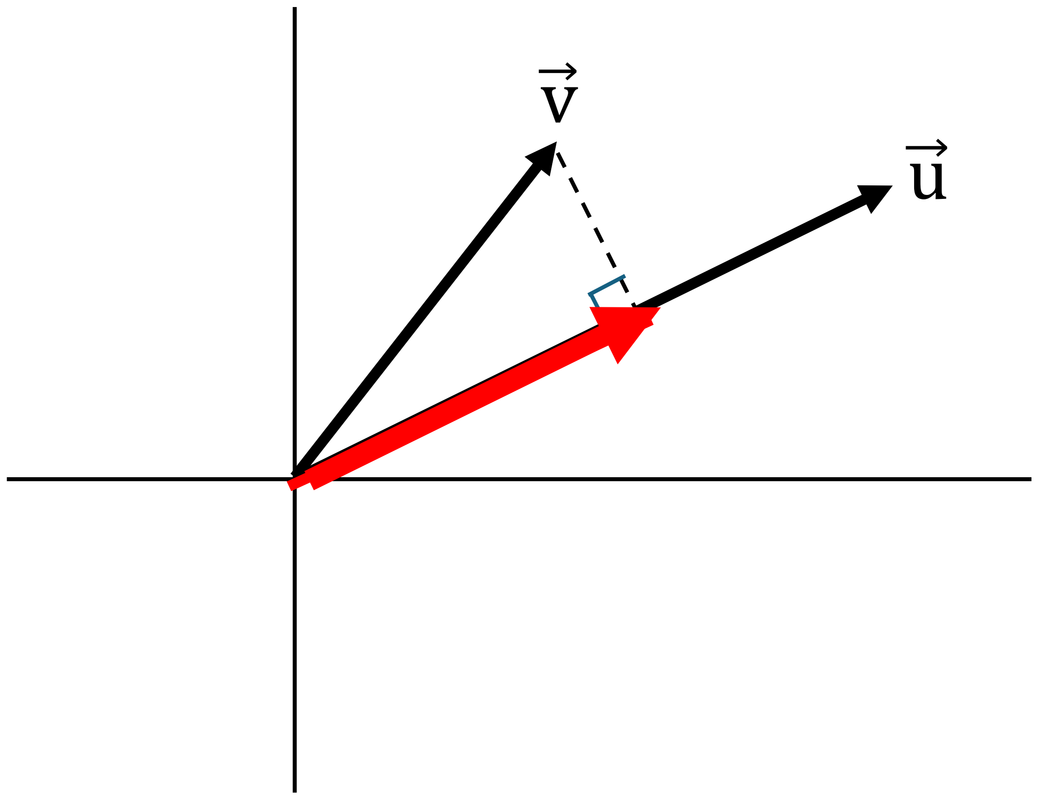
Projection of $\vec{v}$ onto $\vec{u}$

Length: $\dfrac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$

Direction: $\vec{u}$

component of ***v*** in direction of ***u***

$$\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$



Linear projection exercises

v = [2,2]
u = [3,1]

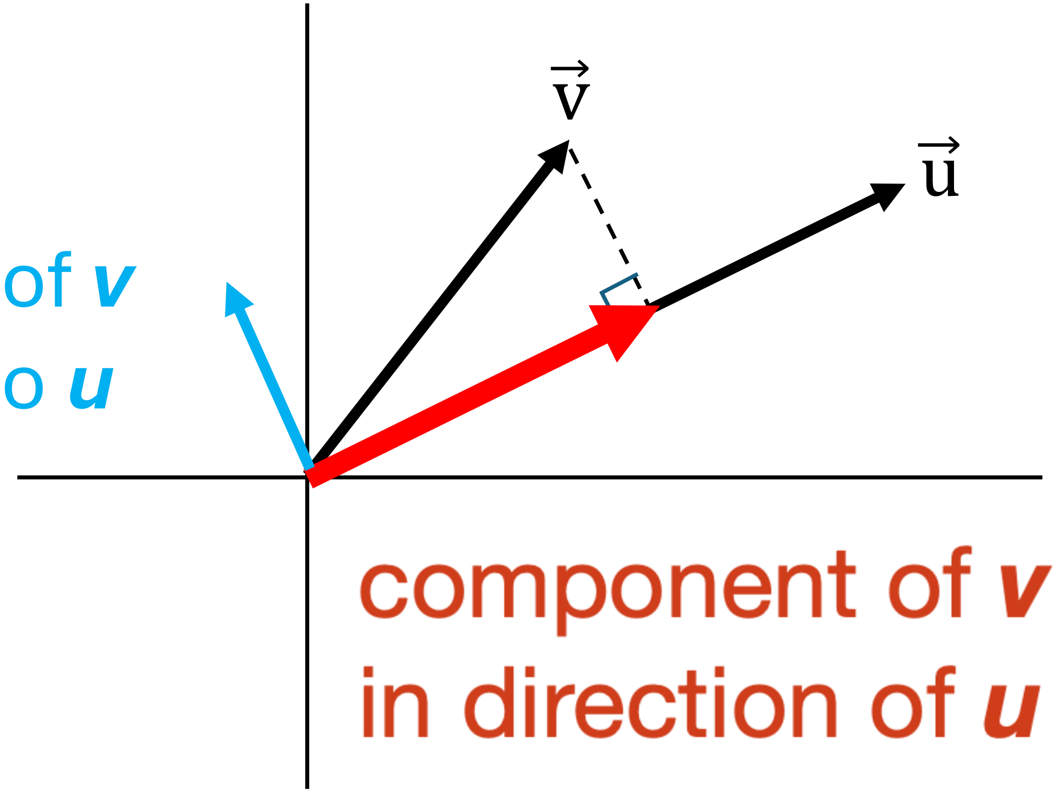
Compute projection of v onto u.

Orthogonality

Two vectors are orthogonal (or *perpendicular*) if their dot product is 0.

$$\vec{w} \cdot \vec{u} = 0$$

component of ***v*** orthogonal to ***u***



$$\vec{w} = \vec{v} - \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

component of ***v*** in direction of ***u***

$$\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

Can decompose any vector ***v*** into its component along ***u*** and its orthogonal component.

Orthogonality exercises

Find a vector orthogonal to x = [1,0] ?

Let

v = [2,2]
u = [3,1]

Find the component of v orthogonal to u.

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